Let \( \omega = \sum_{D \geq k} \omega_D \) be a polynomial differential \( k \)-form written as the sum of its homogeneous components (e.g. \( x_1^{d_1} \cdots x_n^{d_n} dx_1 \wedge \cdots \wedge dx_k \) has homogeneity degree \( D = k + d_1 + \cdots + d_n \)). Put \( E = \sum x_i \partial / \partial x_i \). Then \( L_E \omega_D = D \omega_D \). Since \( L_E \) commutes with the De Rham differential, this shows that \( d \omega = 0 \) implies \( d \omega_D = 0 \) for each \( D \). Since \( L_E = d i_E + i_E d \), we conclude that \( \omega_D = d (i_E \omega_D) / D \), i.e. each \( \omega_D \) is exact, unless \( D = 0 \), i.e. \( \omega \) is a constant function.

Since \( A \) is a closed 1-form, we have for any \( \omega \):
\[
(d - A \wedge)^2 \omega = d^2 \omega - A \wedge d \omega - d(A \wedge \omega) + A \wedge A \wedge \omega = (dA) \wedge \omega = 0.
\]

Thus, \( (\Omega^*(M), D) \) is a complex of sheaves. The sheaves of smooth differential forms are acyclic (as we have shown in class using partitions of unity). Since locally \( A = d \phi \), the differential \( D \) is locally conjugated to the De Rham differential: \( d - A \wedge = e^\phi d e^{-\phi} \). Therefore, by the Poincare lemma, our complex of sheaves is an (acyclic) resolution of the sheaf (let’s denote it \( \mathbb{R} A \)) of smooth functions locally representable in the form form \( \text{Const} \cdot e^\phi \). By the abstract theorem about acyclic resolutions, \( H^*(M; \mathbb{R} A) \) coincides with the cohomology of the “twisted” De Rham complex \( (\Omega^*(M), D) \). Assuming that \( M \) is triangulated, we can obtain (following the construction in Griffiths–Harris) a combinatorial description of the same cohomology. Namely, to each vertex \( v \) of the triangulation, associate the open set \( U_v \) (the “star” of the vertex, which is the interior of all faces of the triangulation containing \( v \) as its vertex). Then all multiple intersections \( U_{v_0} \cap \cdots \cap U_{v_p} \) are contractible (to the simplex \([v_0, \ldots, v_p]\) of the triangulation) or empty (if there is no such face). Thus, non-zero sections \( C e^\phi \) of our sheaf \( \mathbb{R} A \) over each multiple intersection exist and form a 1-dimensional space. By the Leray theorem, the \( \check{\text{C}} \)ech cohomology \( \check{H}(M; \mathbb{R} A) \) can be computed using the cover \( \{U_v\} \). On the other hand, the coboundary operator in the \( \check{\text{C}} \)ech complex built from this cover is dual to the chain complex with the chains spanned by sections \( s_{[v_0, \ldots, v_p]} \) of the sheaf \( \mathbb{R} A \) on the simplexes of the triangulation, and the boundary operator defined by
\[
\partial s_{[v_0, \ldots, v_p]} = \sum_{i=0}^p (-1)^i s_{[v_0, \ldots, \hat{v}_i, \ldots, v_p]}.
\]
3. Adapting the above combinatorial description of Čech cohomology with coefficients in a locally constant sheaf (= local coefficient system), we get the complex $0 \to C \xrightarrow{\delta} C^n \to 0$, where $\delta = \partial^* = [e^{2\pi i a_1} - 1, \ldots, e^{2\pi i a_n} - 1]^T$. Indeed, the space can be retracted to the bouquet of $n$ circles. Equipping the $i$th 1-dimensional cell (parameterized by $[0,1]$) with a single-valued branch of the function $\prod (z - z_i)^{a_i}$ in the role of the section (call it $s_i$) of the sheaf, we find that the boundary of the cell is the 0-cell with the coefficient $s_i(1) - s_i(0) = (e^{2\pi i a_i} - 1)s_i(0)$. This gives the above (co)boundary operator when the basis sections $s_i$ are normalized so that $s_i(0) = 1$. If at least one of $a_i$ is non-integer, the coboundary operator is non-zero, and so the cohomology $H^0 = 0$ and $H^1 \cong \mathbb{C}^{n-1}$.

4. The complement $\mathbb{C}$ to the set of $n$ roots of $z^n = w$ (for a fixed $w$) is homotopy equivalent to the bouquet of $n$ circles. As in the previous problem, the cochain complex computing the cohomology of this complement in our local coefficient system has the form $0 \to \mathbb{C} \to \mathbb{C}^n \to 0$, and the middle coboundary operator maps 1 to $(q - 1, \ldots, q - 1)$, where $q := e^{2\pi i a} \neq 1$. Examine the fibered homotopy $(z, w) \mapsto (e^{2\pi i t/n}z, e^{2\pi i t}w)$. When $t$ varies from 0 to 1, $w$ makes a full turn around 0, the roots of $z^n = w$ experience a cyclic shift, as do the circles in the bouquet, while a single-valued branch of the function $(z^n - w)^a$ is multiplied by $e^{2\pi i t a} = q$. The induced transformation on the space $\mathbb{C}^n$ of 1-cochains is given by the $n \times n$-matrix $(q$ times the cyclic shift$)$:

$$q \times \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots \\ \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}.$$  

The “diagonal” vector $(1, \ldots, 1)$ is invariant under this transformation, which defines therefore the transformation on the quotient $H^1 = \mathbb{C}^n / \mathbb{C}$. This is the generator of the fundamental group’s action on $H^1$. 