Solution to HW4

I was asked to provide solutions to HW4, which you won't get back before the midterm exam, so that you could judge about the correctness of your submitted solutions before the test, and respectively prepare better for it. So, I include my solutions below. Of course, if your solution differs from mine, it does not make yours incorrect (just the same way as when it coincides with mine, it does not make mine correct).

IV.5.2. Describe curves $|f| = const$ and $\arg f = const$ for the function $f(z) = \exp(z^2)$.

`Solution.` For $z = x+iy$, $|\exp(z^2)| = e^{\mathrm{Re}\, z^2} = e^{x^2-y^2}$ and $\arg f(z) = \mathrm{Im}(z^2)+2\pi k = 2xy+2\pi k$, $k \in \mathbb{Z}$. Therefore the level curves $|f| = const$ are the same as $x^2 - y^2 = const$, i.e. hyperbolas with the asymptotes $y = \pm x$, and the level curves $\arg f = const$ are hyperbolas with the asymptotes $x = 0$, $y = 0$ (in both cases — including the union of the pair of the asymptotes themselves as on of the levels).

IV.9.2. Find all values of $\log \log i$.

`Solution.` $\log i = 2\pi i k + i\pi/2$ $k \in \mathbb{Z}$, and hence

$$\log \log i = \ln |2\pi k + \pi/2| \pm i\pi/2 + 2\pi i l, \quad k, l \in \mathbb{Z},$$

where the sign is "+" for $k \geq 0$, and "−" for $k < 0$.

IV.13.4. Let G be the open set obtained by removing from $\mathbb{C}$ the interval $[-1, 1]$ on the real axis. Prove that there is a branch of the function $\sqrt{z^2 - 1}$ in G.

`Solution.` The square root function continuously changes its value to the opposite when its input travels around zero, i.e. in our example when z travels around $+1$ or around -1. The reason why our function has a single-valued branch when the interval $[-1, 1]$ is removed, is that when z travels around this interval, the loop can be interpreted as the composition of two: one going around $+1$, th other around

-1, and so the value of the square root changes its sign twice, and thus returns to its original value.

If this argument does not sound convincing to you, here is a neat solution I've heard from one of the students, which employs the previous exercise from the book. Namely,

$$\sqrt{z^2-1} = \sqrt{\frac{z+1}{z-1}}(z-1).$$

The image of $G = \mathbb{C} - [-1, 1]$ under the linear-fractional transformation $w = (z+1)/(z-1)$ is $\mathbb{C} - [0, -\infty)$, where a branch $\operatorname{Log} w$ of the logarithm is defined. Therefore our function has in G the branch $(1-z)\exp\frac{1}{2}\operatorname{Log}\left((z+1)/(z-1)\right)$.

IV.16.1. Find all the values of $(1+i)^i$.

`Solution.` $\log(1+i) = \ln\sqrt{2} + i\pi/4 + 2\pi i k$, $k \in \mathbb{Z}$. Therefore $(1+i)^i = \exp i \log(1+i) = e^{-\pi(2k+1/4)+i\ln\sqrt{2}} = e^{-\pi(2k+1/4)}\left(\cos\ln\sqrt{2} + i\sin\ln\sqrt{2}\right)$, $k \in \mathbb{Z}$.

IV.16.3. Prove that if f is a branch of z^c in an open set not containing 0, then f is holomorphic and f' is a branch of cz^{c-1}.

`Solution.` Complex differentiability and the value of the derivative are local, i.e. determined by the behavior of the function in any neighborhood of the given point. For any such a point $\neq 0$, z^c the branch of f has the form $e^{c\operatorname{Log} z}$, where $\operatorname{Log} z$ is any local branch of the logarithm. This is a composition of holomorphic functions (since exp and Log are holomorphic), and by the chain rule

$$f'(z) = ce^{c\operatorname{Log} z}\frac{d}{dz}\operatorname{Log} z = ce^{c\operatorname{Log} z}\frac{1}{z} = c\frac{f(z)}{z}.$$

The latter expression is a branch of cz^{c-1} defined in the whole region where the branch of f is defined.