the right hand side of (1) into partial fractions, find an expression for the a_n in terms of z_1 and z_2 and deduce that

$$\rho(S) = \min(|z_1|, |z_2|).$$

(Note that, if $S(X) = S_1(X) \,.\, S_2(X)$, then $\rho(S) \geqslant \min(\rho(S_1), \rho(S_2))$.)

9. Show that, if x, y are real and n is an integer $\geqslant 0$, then

$$\sum_{0 \leqslant p \leqslant n} \sin(px + y) = \sin\left(\frac{n}{2}x + y\right) \sin\frac{n+1}{2}x/\sin\frac{x}{2}.$$
$$\sum_{0 \leqslant p \leqslant n} \cos(px + y) = \cos\left(\frac{n}{2}x + y\right) \sin\frac{n+1}{2}x/\sin\frac{x}{2},$$

(Use $\cos(px + y) + i\sin(px + y) = e^{i(px+y)} = e^{iy}(e^{ix})^p$.)

10. Prove the following inequalities for $z \in \mathbf{C}$:

$$|e^z - 1| \leqslant e^{|z|} - 1 \leqslant |z|\, e^{|z|}.$$

11. Show that, for any integer $n \geqslant 1$ and any complex number z,

$$\left(1 + \frac{z}{n}\right)^n = 1 + z + \sum_{2 \leqslant p \leqslant n} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{p-1}{n}\right) \frac{z^p}{p!},$$

and deduce that

$$e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n.$$

12. Show that the function of a complex variable z defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \left(\text{resp. } \sin z = \frac{e^{iz} - e^{-iz}}{2i}\right)$$

is the analytic extension to the whole plane $\mathbf{C}$ of the function $\cos x$ (resp. $\sin x$) defined in § 3, no. 3, Prove that, for any $z, z' \in \mathbf{C}$,

$$\cos(z + z') = \cos z \cos z' - \sin z \sin z',$$
$$\sin(z + z') = \sin z \cos z' + \cos z \sin z';$$

$$\cos^2 z + \sin^2 z = 1.$$

13. Prove the relations

$$\frac{2}{\pi} x \leqslant \sin x \leqslant x \quad \text{for } x \text{ real and } 0 \leqslant x \leqslant \pi/2.$$