Math 215B. Spring 2020. Homework answers

Homework 2.

1. Since $L^\otimes m \otimes L^\otimes n = L^\otimes (m+n)$, it suffices to show that $c_1(L^\otimes m) = mc_1(L)$, where $L$ is the universal (Hopf) line bundle over $\mathbb{C}P^\infty$, i.e. that $c_1(L^\otimes m) = -m$ for the Hopf bundle $L$ over $\mathbb{C}P^1$. This follows from the fact that $L$ has a meromorphic section with one 1st order pole (and no zeroes), and hence the $m$th power of that section would have one $m$-th order pole (for $m > 0$, and hence $\lfloor m \rfloor$th order zero for $m < 0$).

Another solution (by Jiahao Niu and Junsheng Zhang). Let $L$ be the universal line bundle over $\mathbb{C}P^\infty$. Over $B := \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ consider the line bundle $L' := pr_1^* L \otimes pr_2^* L$, where $pr_i$ are projections to the factors. It suffices to show that $c_1(L') = pr_1^* c_1(L) + pr_2 c_1(L)$. But $H^2(B) = \mathbb{Z} \oplus \mathbb{Z}$ coincides with $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, of the bouquet embedded into the product, over the components of which the bundle $L'$ coincides with $L \otimes \mathbb{C} = L$ and $\mathbb{C} \otimes L = L$ respectively. Thus $c_1(L') = c_1(L) \oplus c_1(L)$ as required.

2. According to the lectures, $p_1 \in H^4(BSO_3)$ maps to $y^2 \in H^4(BSO_2)$. The quaternionic Hopf bundle $V$ over $\mathbb{H}P^\infty$ considered as an $SU_2$-bundle turns into $L \oplus L^*$ when pulled back to $BT^1$, where $L$ is the Hopf bundle over $\mathbb{C}P^\infty$ that pulls-back the universal bundle to its tensor square. Therefore $y^2 \mapsto c_2(V) \mapsto e(V)$, the 1st obstruction to non-vanishing sections of the quaternionic Hopf bundle over $\mathbb{H}P^1$, equals $-1$. Namely, the quaternionic dual bundle has a section, $(q \mapsto q$ in the chart $\mathbb{H}$) with one zero, while the section $1/q = q^*/q^*$ of the Hopf bundle defines the map $S^3 = \{q \in \mathbb{H} \mid |q| = 1\} \rightarrow Sp_1 : q \mapsto q^*$ of degree $-1$ ($(a + bi + cj + dk)^* = a - bi - cj - dk$ is orientation-reversing). Thus $u \mapsto x^2$. The double-cover map $T^1 \rightarrow SO_2 \equiv T^1$ yields $BT^1 \rightarrow BT^1$ which pulls-back the universal bundle to its tensor square. Therefore $y^2 \mapsto 2x$ (by Problem 1). Thus $y^2 \mapsto 4x^2$, and so $p_1 \mapsto 4u$.

3. Exact homotopy sequences of the bundles $\mathbb{H}V(\infty, n) \rightarrow \mathbb{H}V(\infty, 1) \rightarrow S^{4n-1}$, where $\mathbb{H}V(\infty, n - 1)$ (mapping an $n$-frame to its 1st vector) show by induction on $n$, starting with $S^\infty \sim pt$, that $\pi_q \mathbb{H}V(\infty, n) = 0$ for all $q$. According to the classification theory of principal bundles, this exactly means that the principal $Sp_n$-bundle $\mathbb{H}V(\infty, n) \rightarrow \mathbb{H}G(\infty, n)$ is universal, and $\mathbb{H}G(\infty, n) = BSp_n$. 

1
4. In the quaternionic Gaussian row-reduction algorithm, one should be careful to multiply rows by scalars always on the same side (on the right under our convention). With this case in place, one obtains the usual way the decomposition of $\mathbb{H}G(\infty, n)$ into Schubert cells of dimensions multiple to 4, with the number of cells of dimension $4m$ (and hence the rank of $H^{4m}(\mathbb{H}G(\infty, n))$) equal to the number of partitions of $m$ with at most $n$ summands. This count agrees with the number of monomials $\sigma_1^{d_1} \ldots \sigma_n^{d_n}$ of degree $1d_1 + 2d_2 + \cdots + nd_n = m$ in the ring of symmetric polynomials. Therefore it suffices to show that the map $BSp_1^n \to BSp_n$ induces a homomorphism $H^*(BSp_n) \to H^*(BSp_1^n) = \mathbb{Z}[u_1, \ldots, u_n]$ surjective onto the subring of symmetric polynomials. The fact that the image is $S_n$-invariant is obvious from the invariance of the sum of $n$ quaternionic line bundles to the permutation of the summands. Rather than mimicking further the obstruction theory approach used in the complex case, let’s interpret the universal $Sp_n$ bundle with the fiber $\mathbb{H}^n$ as the complex bundle $V$ with the fiber $\mathbb{C}^{2n}$ and lift all classes to $BT^n \mapsto BSp_1^n = BSp_n$ (referring to Problem 2). The total Chern class $c(V) \in H^*(BSp_n)$ is mapped to $(1 - x_1)(1 + x_1) \cdots (1 - x_n)(1 + x_n) \in H^*(BT^n)$, which is $\sum_{m=0}^{n}(-1)^m \sigma_m(x_1^2, \ldots, x_n^2)$. Since (according to Problem 2) $u_i \mapsto x_i^2$, we conclude that $c_{2m}(V) \mapsto (-1)^m \sigma_m(u_1, \ldots, u_n)$. Therefore $H^*(BSp_n)$ is a free polynomial algebra of $c_2(V), \ldots, c_{2m}$ (while all $c_{odd}(V) = 0$), and is identified with the algebra of symmetric polynomials in $u_1, \ldots, u_n$. 

Therefore $H^*(BSp_n)$ is a free polynomial algebra of $c_2(V), \ldots, c_{2m}$ (while all $c_{odd}(V) = 0$), and is identified with the algebra of symmetric polynomials in $u_1, \ldots, u_n$. 

Therefore $H^*(BSp_n)$ is a free polynomial algebra of $c_2(V), \ldots, c_{2m}$ (while all $c_{odd}(V) = 0$), and is identified with the algebra of symmetric polynomials in $u_1, \ldots, u_n$. 

Therefore $H^*(BSp_n)$ is a free polynomial algebra of $c_2(V), \ldots, c_{2m}$ (while all $c_{odd}(V) = 0$), and is identified with the algebra of symmetric polynomials in $u_1, \ldots, u_n$.