**Math 214: Differential Manifolds**

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HOMEWORK 1

**1.** Consider the stereographic projections of the sphere $x^2+y^2+z^2 = 1$ from the North Pole $(0,0,1)$ and from the South Pole $(0,0,-1)$ onto the plane $z = 0$ as two charts, compute the transition maps between the charts, and show that they are smooth.

**2.** Find out which of the following manifolds are diffeomorphic to each other (indicate the map when they are):

(a) the real projective space $\mathbf{R}P^3$;

(b) the intersection of the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in $\mathbf{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$;

(c) the manifold $T_1S^2$ of unit tangent vectors to a 2-sphere;

(d) the *configuration space* ( i.e. the space of all possible positions) of a (sufficiently general) rigid body in the 3-space, fastened at one point;

(e) the Stiefel manifold $V(3,2)$ of 2-frames in $\mathbf{R}^3$ (i.e. of ordered pairs of unit perpendicular vectors);

(f) the group $SO_3$ of rotations of $\mathbf{R}^3$.

**3.** State Implicit Function Theorem, Inverse Function Theorem, and derive each one of them from the other.

HOMEWORK 2

**1.** Show that the following surgeries on connected 2-dimensional manifolds are equivalent, i.e. produce homeomorphic manifolds:

(a) In the presence of a Möbius strip embedded into the surface, attaching a cylinder in the orientation-respecting fashion is equivalent to attaching a cylinder in the disorienting fashion.

(b) Attaching two Möbius strips is equivalent to attaching a cylinder in a disorienting fashion.

(c) Convince yourself that any surface obtained from the sphere by the surgeries of attaching cylinders and/or Möbius strips is homeomorphic to one of the following: $S_g^2$ (the sphere with $g$ handles), $P_g^2$ (the projective plane with $g$ handles), $K_g^2$ (the Klein bottle with $g$ handles), $g = 0, 1, 2, ...$

**2.** Show that any maximal ideal in the algebra $C^\infty(M)$ of smooth functions on a manifold $M$ is the maximal ideal $\mathfrak{m}_x$ of some point $x \in M$, i.e. consists of all smooth functions vanishing at $x$. Deduce from this that any algebra homomorphism $C^\infty(M) \to C^\infty(N)$ is induced by a smooth map $N \to M$. (For simplicity, you may assume in this problem that $M$ is compact.)

**3.** Let $W$ be a real vector space of finite dimension, and let $G := Gr_k(W)$ denote the Grassmann manifold of all $k$-dimensional vector subspaces in $W$. Consider such a subspace $V \subset W$ as a point $v \in G$, and show that the tangent space $T_v G$ can be *canonically* identified with the space of linear maps $Hom(V, W/V)$.

HOMEWORK 3

**1.** A hunter is positioned at the origin of the coordinate plane, and at all other integer points of the plane are occupied by identical rabbits. Show that however small are the rabbits, the hunter cannot miss no matter what direction he shoots.

**2.** Consider a smooth free action of a compact Lie groups $G$ on a smooth manifold $M$. Equip the quotient space $M/G$ with the structure of a smooth manifold such that the canonical projection $\pi : M \to M/G$ is a submersion. Prove that $\pi^*(C^\infty(M/G)) = C^\infty(M)^G$ (the subalgebra of $G$-invariant functions).

**3.** The map $S^{2n-1} \to \mathbf{C}P^{n-1}$, defined by associating to a unit vector in $\mathbf{C}^n$, the complex line spanned by it, is called a *Hopf bundle*. Show that a Hopf bundle is a locally trivial bundle with the fiber $S^1$. For $n = 2$, sketch the way the fibers of the Hopf bundle $S^3 \to S^2$ fill in the space $\mathbf{R}^3$ obtained from $S^3$ by a stereographic projection.

HOMEWORK 4

**1.** Prove that a (non-compact) $n$-dimensional submanifold in a euclidean space can be smoothly embedded into $\mathbf{R}^{2n+1}$ as a closed subset.

**2.** Prove that a compact $n$-dimensional manifold with boundary can be embedded into the half-space $\mathbf{R}^{2n+1}_+$ so that the boundary of the manifold lies in the boundary of the half-space, the interior of the manifold lies in the interior of the half-space, and the tangent spaces to the manifold at the points on the boundary are perpendicular to the boundary of the half-space.

**3.** $(a)$ Compute pairwise Lie brackets of the basis

$$\{x^m \partial/\partial x,\ m = 0, 1, 2, \dots\}$$

in the Lie algebra of polynomial vector fields on the line, and
$(b)$ identify inside it a Lie subalgebra isomorphic to the Lie algebra of traceless $2 \times 2$-matrices equipped with the matrix commutator operation $[A, B] := AB - BA$.

Could you think of any *a priori* explanation of the isomorphism?

## HOMEWORK 5

**1.** Let $v$ and $w$ be two vector fields on a manifold $M$, and let $g_t : M \to M$ be a family of diffeomorphisms such that $g_0 = id_M$, and $d/dt|_{t=0} g_t x = v(x)$ for all $x \in M$ (e.g. $g_t$ can be the flow $g^t$ of $v$). Compute the *Lie derivative* of $w$ along $v$, defined by:

$$L_v w := \frac{d}{dt}|_{t=0}(g_t)_* w.$$

**2.** Given a bilinear form $\mathcal{A}$ on a vector space $V$, denote by $\mathrm{Aut}(V, \mathcal{A})$ the Lie group of linear transformations on $V$ preserving $\mathcal{A}$, and by $\mathrm{aut}(V, \mathcal{A})$ the corresponding Lie algebra.

Let $S : V \to V^*$ and $A : V^* \to V$ be two linear transformations between dual spaces, and suppose that $S^* = S$, and $A^* = -A$. Prove that

$$AS \in \mathrm{aut}(V, \mathcal{S}), \text{ where } S(x, y) = \langle Sx, y \rangle,$$

and

$$SA \in \mathrm{aut}(V^*, \mathcal{A}), \text{ where } A(u, v) = \langle u, Av \rangle.$$

**3.** Prove that the exponential map $\exp : \mathfrak{g} \to G$ provides a diffeomorphism between sufficiently small neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$. Compute the range of the exponential map for $G = GL_n(\mathbf{C})$. Give an example of a connected Lie group $G$ whose exponential map is not surjective.

## HOMEWORK 6

**1.** Find the maximal dimension of integral submanifolds of the 3-dimensional distributions in $\mathbf{R}^4 - 0$ given by the equation:

$$(a)\ \ x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 = 0;$$

$$(b)\ \ x_1 dy_1 + y_1 dx_1 + x_2 dy_2 + y_2 dx_2 = 0.$$

**2.** Study the Cartan distribution in the space where the graphs of functions ($u$) in one variable ($x$) together with their derivatives up to order $r$ "live." What is the dimension of the space? Of the Cartan distribution? Describe the distribution explicitly. To a smooth function $u = u(x)$, associate an integral curve of the distribution. Is the distribution integrable? Does it have 2-dimensional integral submanifolds?

**3.** Find a necessary and sufficient condition for a left-invariant distribution on a Lie group to be integrable. Apply your criterion to the 2-dimensional distribution on $S^3 = \{|z|^2 + |w|^2 = 1\}$ defined by orthogonal complements to the fibers of the Hopf bundle.

## HOMEWORK 7

**1.** Let $E = \mathbf{R} \times M \to M$ be the trivial $\mathbf{R}$-bundle over a simply-connected $n$-dimensional manifold $M$, and let $\Pi$ be an $n$-dimensional distribution on $E$ which is everywhere transverse to the fibers of the bundle and is invariant with respect to translations in the direction of $\mathbf{R}$. Prove that a global section $M \to E$ of the bundle integral to the distribution exists if and only if the distribution is involutive.

**2.** Which of the Grassmann manifolds $Gr_{k,n}(\mathbf{R})$ are simply-connected?

**3.** Classify up to isomorphism all connected Lie groups with the Lie algebra $so_4(\mathbf{R})$.

## HOMEWORK 8

**1.** Let $A$ and $B$ be two anti-symmetric $4 \times 4$-matrices. For how many values of $t$ can the matrix $A + tB$ degenerate?

**2.** *Plücker embeddings of grassmannians.* An exterior $k$-form $\omega \in \Lambda^k V^*$ on $V$ is called *decomposable* if it is an exterior product of $k$ linear forms. Prove that in the projective space $Proj(\Lambda^k V^*)$, the locus defined by non-zero decomposable $k$-forms is isomorphic to the Grassmann manifold of $k$-dimensional subspaces in $V^*$.

**3.** Prove that a non-degenerate quadratic hypersurface in $\mathbf{C}P^5$ (i.e. a hypersurface defined in $Proj(\mathbf{C}^6)$ by a homogeneous equation $Q(z_0, ..., z_5) = 0$ written in linear coordinates $(z_0, ..., z_5)$ on $\mathbf{C}^6$ and corresponding to a non-degenerate quadratic form $Q$) is diffeomorphic (in fact isomorphic as a complex algebraic manifold) to the Grassmannian $Gr_{2,4}(\mathbf{C})$.

## HOMEWORK 9

**1.** An $n$-dimensional manifold is called orientable if it admits an atlas all of whose transition functions have positive Jacobians. Show that an $n$-dimensional manifold is orientable if and only if it admits a nowhere vanishing differential $n$-form (such a form is called a *volume* form).

**2.** Let $\omega^n$ be a volume form on a manifold $M$.

(a) Show that the *contraction* operation

$$Vect(M) \to \Omega^{n-1}(M) : \quad v \mapsto i_v \omega$$

(defined by substituting $v$ into $\omega$ as the 1st argument) defines an isomorphism of $C^\infty(M)$-modules.

(b) Characterize those vector fields whose flows preserve $\omega$, in terms of the differential $n-1$-forms corresponding to them under the contraction isomorphism.

**3.** Let $\alpha$ be a non-vanishing differential 1-form on an $n$-dimensional manifold. Prove that the $n-1$-dimensional distribution $\ker\alpha$ (i.e. a "Pfaff equation") is integrable if and only $\alpha \wedge d\alpha = 0$.

## HOMEWORK 10

**1.** Prove that $H^\bullet_{DR}(T^n) = \left(\Omega^\bullet(T^n)\right)^{T^n}$ (i.e. the space of translation invariant differential forms on the torus).

**2.** Show that any complex manifold, if considered as a real manifold, is orientable.

**3.** The equation $F(z_1, z_2) = 0$, where $F$ is a polynomial of degree $d$, defines an algebraic degree $d$ curve in $\mathbf{C}^2$. Let us assume for simplicity that the complex curve is non-singular, and orient the curve considered as a real surface by requiring that the direction of the $90°$ rotations in its tangent planes, defined as the multiplication by $\sqrt{-1}$, is called counter-clockwise.

Let $C_r$ be the part of this curve inside the ball $|z_1|^2 + |z_2|^2 \leq r^2$ of radius $r$. Compute the limit:

$$\lim_{r\to\infty} \frac{1}{r^2} \int_{C_r} dx_1 \wedge dy_1 + dx_2 \wedge dy_2,$$

where $x_i$ and $y_i$ are the real and imaginary parts of $z_i$ respectively.

## HOMEWORK 11

**1.** Among smooth curves $t \mapsto (x(t), y(t)) \in \mathbf{R}^2$, identify extrema of the functional ($k$ is a constant parameter)

$$\int \left[\dot{x}^2 + \dot{y}^2 + k(x\dot{y} - y\dot{x})\right]\ dt.$$

**2.** Compute explicitly the Laplace operator $\Delta = dd^* + d^*d$ on the space $\Omega^*_c(\mathbf{R}^3)$ of compactly supported differential forms in $\mathbf{R}^3$, equipped with the inner product

$$(\alpha, \beta) = \int_{\mathbf{R}^3} \alpha \wedge (\star\beta),$$

where $\star$ is Hodge's star-operator.

**3.** On a connected Riemann manifold, define the distance between any two points to be the infimum of arc lengths over all (piecewise) smooth curves connecting the points. Prove that the manifold, equipped with this distance, is a metric space (complete if compact). Deduce that any (connected) manifold (not necessarily compact) can be equpped with the structure of a complete metric space.

# HOMEWORK 12

**1.** (The tubular neighborhood theorem via exponential maps for geodesics.) Let $\subset M$ be a compact submanifold in a Riemann manifold. Given $x \in N$, to a non-zero vector $v \in T_x M$ normal to $N$, associate a point in $M$, defined as the endpoint of the geodesic curve of length $|v|$ issued from $x$ in the direction of the vector $v/|v|$. Prove that this construction provides a diffeomorphism between a neighborhood of $N$ in $M$ and a neighborhood the zero section in the normal bundle of $N$.

**2.** Construct a $C^1$-function $f : [0,1] \to [0,1]$ whose critical value set coincides with the standard Cantor set. Show that the critical value set of the $C^1$-function $g : [0,1] \times [0,1] \to [0,2]$, defined by $g(x,y) = f(x) + f(y)$ has positive measure.

**3.** Prove that on any manifold, functions with only non-degenerate critical points form a massive subset in $C^\infty(M)$.