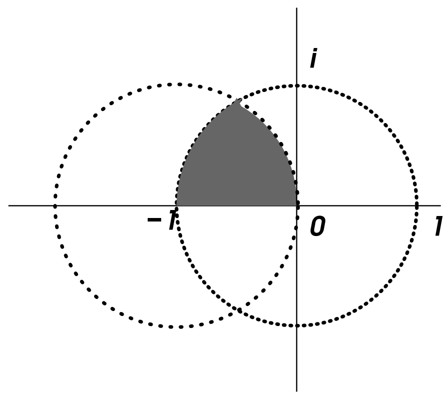
Answers to Homework Exercises

HW1

1. From $z^2 - 2z + \sqrt{3}i = 0$, using the "quadratic formula", we find

$$z = 1 \pm \sqrt{1 - \sqrt{3}i} = 1 \pm \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = 1 \pm \sqrt{\frac{3}{2}} \mp \frac{i}{\sqrt{2}}.$$

2. The set given by $|z| < 1, |z+1| < 1, \mathrm{Re}(iz) < 0$ is the intersection of two open disks of radius 1 centered at $z = 0$ and $z = -1$ respectively with each other and with the open upper half-plane:



3.

$$\tan X = \frac{\sin X}{\cos X} = \frac{X - X^3/6 + X^5/120 + \cdots}{1 - (X^2/2 - X^4/24 + \cdots)}$$

$$= X\left(1 - \frac{X^2}{6} + \frac{X^4}{120} + \cdots\right)\left(1 + \left(\frac{X^2}{2} - \frac{X^4}{24} + \cdots\right) + \left(\frac{X^2}{2} + \cdots\right)^2 + \cdots\right)$$

$$= X + X^3\left(\frac{1}{2} - \frac{1}{6}\right) + X^5\left(\frac{1}{120} - \frac{1}{12} - \frac{1}{24} + \frac{1}{4}\right) + \cdots$$

$$= X + \frac{X^3}{3} + \frac{2X^5}{15} + \cdots =: Y.$$

For $X = \arctan Y$ solve $X = Y - X^3/3 - 2X^5/15 + \cdots$ by iterations:

$$X = 0 \text{ (on the right)} \;\Rightarrow\; X = Y + \cdots \text{ (on the left)}$$

$$X = Y + \cdots \;\Rightarrow\; X = Y - Y^3/3 + \cdots$$

$$X = Y - \frac{Y^3}{3} + \cdots \;\Rightarrow\; X = Y - \frac{(Y - Y^3/3)^3}{3} - \frac{2Y^5}{15} + \cdots,$$

i.e. $\arctan Y = Y - Y^3/3 + Y^5/5 + \cdots$ Alternatively, this can be found by termwise anti-differentiation from $(\arctan Y)' = 1/(1+Y^2) = 1 - Y^2 + Y^4 - \cdots$.

4. The derivative of order $p-1$ of $1/(1-X)$ equals $(p-1)!/(1-X)^p$. This is true for functions, but also follows (from the product rule) by induction on p for the formal power series. Therefore

$$\left(\sum_{n\geq 0} X^n\right)^p = \frac{1}{(p-1)!}\frac{d^{p-1}}{dX^{p-1}}\left(\sum_{n\geq 0} X^n\right) = \sum_{m\geq 0}\binom{m+p-1}{p-1}X^m.$$

HW2

I.4. (a) $\sum q^{n^2}z^n$ has infinite convergence radius since $|q^{n^2}|^{1/n} = |q|^n \to\to 0$ when $|q| < 1$. (b) $\sum n^p z^n$ has convergence radius 1 since $(n^p)^{1/n} = (n^{1/n})^p \to 1^p = 1$. (c) is essentially the sum of two geometric series with convergence radii $1/a$ and $1/b$. Therefore $|a_n|^{1/n}$ consists of two convergent sequences with the limits a and b, and so $\rho = 1/\max(a,b) = \min(1/a, 1/b)$.

I.8 bc. Indeed, from $(1-\alpha z-\beta z^2)(\sum a_n z^n) = z$ we have $a_n - \alpha a_{n-1} - \beta a_{n-2} = 0$ for $n \geq 2$, while $a_0 = 0$ and $a_1 = 1$. Thus, within the convergence radius of series $S = \sum a_n z^n$, the identity holds for the functions:

$$S(z) = \frac{z}{1-\alpha z-\beta z^2} = \frac{z_1 z_2}{z_2 - z_1}\left[\frac{1}{1-z/z_1} - \frac{1}{1-z/z_2}\right],$$

where we assume $z_1 \neq z_2$. Therefore $a_n = (z_1^{-n} - z_2^{-n})/(z_1^{-1} - z_2^{-1})$. As in I.4(c), the radius of convergence equals $\min(|z_1|, |z_2|)$. When $z_1 = z_2 =: z_0$, $S(z) = z/(1-z/z_0)^2 = \sum_{n\geq 0} n z_0^{-n} z^n$ (as in Problem 4 from HW1 with $p = 2$), which has convergence radius $|z_0|$ (since $n^{1/n} \to 1$ as $n \to \infty$), and implies $a_n = n/|z_0|^n$.

I.11. From the binomial formula, we have

$$\left(1+\frac{z}{n}\right)^n = \sum_{p=0}^{n}\frac{n!}{(n-p)!p!}\frac{z^p}{n^p} = \sum_{p=0}^{n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{p-1}{n}\right)\frac{z^p}{p!}.$$

For every $\epsilon > 0$ and every z, there exists N large enough so that $\sum_{p>N}|z|^p/p! < \epsilon/3$, and hence $\sum_{p=N+1}^{n}(1-1/n)\cdots(1-(p-1)/n)|z|^n/p! < \epsilon/3$. For such N,

$$\left|e^z - \left(1+\frac{z}{n}\right)^n\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \sum_{p=0}^{N}\left[1-\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{p-1}{n}\right)\right]\frac{|z|^p}{p!}.$$

The last expression is a polynomial in $|z|$ of fixed degree N whose coefficients tend to 0 as $n \to \infty$, and so the expression becomes smaller than $\epsilon/3$ for n large enough.

I.16. (i) Indeed, as suggested,

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = \alpha_1(\beta_1 - \beta_2) + (\alpha_1+\alpha_2)(\beta_2-\beta_3)$$
$$+ (\alpha_1+\alpha_2+\alpha_3)(\beta_3-\beta_4) + \cdots + (\alpha_1+\cdots+\alpha_n)\beta_n.$$

Therefore, since $|\sum_{i=1}^{k} \alpha_i| \leq M$ for all k, and $\beta_1 \geq \beta_2 \geq \cdots \beta_n \geq 0$, we have

$$\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \leq M[(\beta_1 - \beta_2) + (\beta_2 - \beta_3) + \cdots + \beta_n] \leq M\beta_1.$$

(ii) Given $\epsilon > 0$ pick m large enough so that $M := \sup_n |a_{m+1} + \cdots + a_n| < \epsilon$. Such m exists since the series $\sum_i a_i$ is assumed to converge), and hence its partial sums form a Cauchy sequence. Take $\alpha_i = a_{m+i}$, $\beta_i = x^{m+i}$, where $0 \leq x \leq 1$. Then by (i)

$$\left|\sum_{i=m+1}^{m+n} a_i x^i\right| \leq M x^{n+1} < \epsilon.$$

Therefore polynomials $s_n(x) := \sum_{i=0}^{n} a_n x^n$ form a Cauchy sequence in the complete metric space of continuous functions on $[0,1]$ with respect to the norm $\|s_m - s_n\| = \max_{0 \leq x \leq 1} |s_m(x) - s_n(x)|$ of uniform convergence. Therefore $s(x) := \sum_i a_i x^i$ is a uniform limit of $\{s_n\}$, and is therefore a continuous function on $[0,1]$. The continuity means that $\lim_{x \to 1^-} s(x) = s(1) = \sum_i a_i$.

Finally, we can apply (ii) to $\sum_i (-1)^{k-1} x^k / k$ (since $\sum (-1)^{k-1}/k$ converges). For $|x| < 1$, this is the power series expansion of $\log(1+x)$. Therefore $\sum (-1)^{k-1}/k = \lim_{x \to 1^-} \log(1+x) = \log 2$ since $\log(1+x)$ is continuous at $x = 1$.

HW3

1. $\tanh z := (e^z - e^{-z})/(e^z + e^{-z}) = (e^{2z} - 1)/(e^{2z} + 1)$ is πi-periodic (since $e2z$ is πi-periodic). Its zeroes are found from $e^{2z} = 1$, i.e. $z = \pi i n$, $n \in \mathbb{Z}$. In particular, $\tanh z$ has no other periods except integer multiples of πi (since the set of zeroes must be invariant under translations by periods of the function). The poles of $\tanh z$ are found from $e^{2z} = -1$, i.e. $z = \pi i(n + 1/2)$, $n \in \mathbb{Z}$. By definition, $\sin z := (e^{iz} - e^{-iz})/2i$, $\cos z := e^{iz} + e^{-iz})/2$, and hence $\tanh z = i \tan iz$. From Problem 3 of HW1, $\tan X$ expands near $X = 0$ as $X + X^3/3 + 2X^5/15 + \cdots$. Therefore near $z = -\pi i$ we have:

$$\begin{aligned}\tanh z &= -i \cdot i(z + \pi i) - \frac{i}{3} i^3 (z + \pi i)^3 - \frac{2i}{15} i^5 (z + \pi i)^5 + \cdots \\ &= (z + \pi i) - \frac{1}{3}(z + \pi i)^3 + \frac{2}{15}(z + \pi i)^5 + \cdots\end{aligned}$$

2. $\sin 2\pi z$ is analytic in $\mathbb{C}$ and has zeroes at $z = n = 0, \pm 1, \pm 2, \ldots$. Thus $1/\sin(2\pi/z)$ has poles at $z = 1/n$, and can be considered as a meromorphic function in the half-plane $\operatorname{Re} z > 0$ (or, alternatively, in $\mathbb{C} - \{0\}$).

II.6. Parameterizing the ellipse E as $z(t) = a \cos t + ib \sin t$, we have $\dot{z}(t) = -a \sin t + ib \cos t$, and $\dot{z}\bar{z} = (b^2 - a^2) \sin t \cos t + iab$. Therefore

$$2\pi i = \oint_E \frac{dz}{z} = i \operatorname{Im} \int_0^{2\pi} \frac{\dot{z} dt}{z} = i \operatorname{Im} \int_0^{2\pi} \frac{\dot{z}\bar{z}\ dt}{z\bar{z}} = \int_0^{2\pi} \frac{iab\ dt}{a^2 \cos^2 t + b^2 \sin^2 t}.$$

II.8. If $f = u + iv$ is holomorphic, then $(a - bi)f$ is holomorphic too, and $au + bv$ is its real part. If it is constant $(= c)$ then the Cauchy-Riemann equations imply that (in a connected domain), the imaginary part is also constant, i.e. $(a - bi)f = const$. Since $a - bi \neq 0$ (for otherwise $c = 0$ too in contradiction with the hypothesis), we conclude that $f = const$.

HW4

II.3. Since ϕ is holomorphic, it is differentiable and hence continuous. Therefore $\Gamma := \phi \circ \gamma$ is continuous when $\gamma : t \mapsto z(t) = x(t) + iy(t)$ is. On each interval of differentiability, this composition of differentiable functions is differentiable, but it is not obvious from the definition of holomorphy that the derivative $d\phi(\gamma(t))/dt = \phi_z \dot{\gamma} + \phi_{\bar{z}} \dot{\bar{\gamma}}$ is continuous. (In fact the 2nd summand is zero due to the Cauchy-Riemann equations for ϕ, while ϕ_z in the first summand coincides with ϕ'.) However, by definition of differentiable paths, $\dot{x}$ and $\dot{y}$ are continuous, while ϕ is in fact infinitely differentiable due to the established analyticity of holomorphic functions. Finally,

$$\begin{aligned}\int_\Gamma f(w)\ dw &:= \int_a^b f(\phi(\gamma(t))\frac{d\phi(\gamma(t))}{dt}dt \\ &= \int_a^b f(\phi(\gamma(t))\ \phi'(\gamma(t))\ dt =: \int_\gamma f(\phi(z))\ \phi'(z)\ dz.\end{aligned}$$

II.7. We have $|t^n| = R^n$ when $|t| = R$, while $P(t)/t^n - 1 = a_1/t + a_2/t^2 + \cdots + a_n/t^n \to 0$ as $R \to \infty$. Therefore for R sufficiently large, $|P(t) - t^n| < |t^n|$, and consequently for $0 \le \epsilon \le 1$, we have $|t^n + \epsilon(P(t) - t^n)| > |t^n|/2 > 0$. When ϵ varies from 0 to 1, we obtain a homotopy between the loop $z = t^n|_{|t|=R}$ and $z = P(t)|_{|t|=R}$ which lies in $\mathbb{C} - \{0\}$ for R large enough. Thus, for such R, the loops have the same index relative to $z = 0$, while the index is equal to n for $z = t^n|_{|t|=R}$ (for any $R > 0$): $\oint dz/z = n \oint dt/t = 2\pi i n$.

On the other hand, if P doesn't vanish anywhere, $\oint \frac{dP(t)}{P(t)} = \oint \frac{P'(t)\ dt}{P(t)} = 0$ for any closed integration path, because the plane is simply-connected. This contradiction ($2\pi i n \neq 0$ for $n \geq 1$) proves the Fundamental Theorem of Algebra: a positive degree polynomial has a complex root.

II.10. When f is holomorphic in a domain D, define $\bar{f}(z) := \overline{f(\bar{z})}$ (in the reflected domain $\bar{D}$, which coincides with D when D is symmetric). Then $\bar{f}$ is also holomorphic in D. Indeed, under the complex conjugation (reflection) of both the domain and target planes of f, the linearization $w \mapsto f'(z)w$ of f (at some point z) becomes $w \mapsto \overline{f'(z)\bar{w}} = \overline{f'(z)}w$, i.e. remains the multiplication by a complex number (and moreover, $\bar{f}'(\bar{z}) = \overline{f'(z)}$.) On the real axis, $f(x) = g(x) + ih(x), \bar{f}(x) = g(x) - ih(x)$, where the real-valued functions $g, h : I = D \cap \mathbb{R} \to \mathbb{R}$ are the real and imaginary parts of $f|_I$. Since I is assumed to be non-empty (and open in $\mathbb{R}$), and D connected, the functions $g(z) := [f(z)+\bar{f}(z)]/2$ and $h(z) := [f(z)-\bar{f}(z)]/2i$ are holomorphic in D, and are uniquely determined (via analytic continuation) by their real-valued restrictions to I. In particular, $\bar{g}(z) = g(z)$ and $\bar{h}(z) = h(z)$ for all $z \in D$, since this is true for all $z \in I$.

II.11. Since $(\log f)' = f'/f$ and $(\log g)' = g'/g$, we have

$$f(z) = \exp \int_a^z \frac{f'(t)}{f(t)} dt, \quad g(z) = \exp \int_a^z \frac{g'(t)}{g(t)} dt,$$

where the integrals are taken along any piece-wise differentiable path in D connecting a with z. (In other words, $f(z) = e^{\log f(z)}$ where the ambiguity of the logarithm is compensates by the periodicity of the exponential function.) Since $f'/f - g'/g$ has a sequence of zeroes a_n converging to a, we conclude that a is a non-isolated zero, and hence $f'/f = g'/g$ on D. Thus, f and g coincide up to the multiplicative "integration constant" i.e. if $f(a) = c(g(a)$, then $f(z) = cg(z)$ for all $z \in D$.

Alternatively (as some students suggested during my office hours), instead of integration, one can differentiate the (already established) identity $f'/f = g'/g$, and conclude inductively that, if $f(a) = cg(a)$, then $f^{(n)}(a) = cg^{(n)}(a)$ for all n. Then f and cg have the same Taylor series at a, and therefore (by analytical continuation) coincide in the entire connected domain D.

HW5

1. Arguing *ad absurdum*, assume that for every $N = 1, 2, 3, ...$ there is an $1/N \times 1/N$ square $\square_N$ (one of the N^2 of those) such that $F(\square_N)$ is not contained in any of the neighborhoods $U_x \subset D$ where the property P holds. Due to compactness of $[0, 1]^2$, the sequence of the centers v_N of the squares $\square_N$ has a convergent subsequence v_{N_k}. Let v be its limit. Then $V := F^{-1}(U_{F(v)})$ is an open subset in $[0, 1]^2$, and hence for k large enough the $1/N_k \times 1/N_k$-square $\square_{N_k}$ lies in V. (Indeed, for some $\epsilon > 0$, V contains all points of $[0, 1]^2$ of distance $< \epsilon$ to v, and hence, when k is large enough so that $|v_{N_k} - v| < \epsilon/2$, and $1/2^{N_k} < \epsilon/\sqrt{2}$, all points of $\square_{N_k}$ lie within the distance ϵ from v.) Thus, $F(\square_{N_k}) \subset U_{F(v)}$ in contradiction with the hypothesis.

Alternatively, for each $u \in [0, 1]^2$, let V_u be a open disk centered at u such that P holds in $F(V_u)$, and let ϵ_u be its radius. From the cover of $[0, 1]^2$ by open disks of the radiuses $\epsilon_u/2$ centered at u, pick a finite subcover (which exists due to compactness of the square) by such disks (let they be centered at $u_1, \ldots, u_n$), and put $\epsilon := \min(\epsilon_{u_1}, \ldots, \epsilon_{u_n})$. Then for N such that $1/N < \epsilon/2\sqrt{2}$, any $1/N \times 1/N$-square is contained in one of V_{u_i} (because its center lies within the distance $\epsilon/2$ from one of u_i, $i = 1, \ldots, n$.)

2. *Answer:* $w = e^{i\theta}(z - z_0)/(1 - \bar{z}_0 z)$ where $|z_0| < 1$.

Indeed, this transformation is invertible, maps the unit circle $|z| = 1$ (in which case $1/z = \bar{z}$) to the unit circle:

$$|w| = |z| \frac{|1 - z_0/z|}{|1 - \bar{z}z|} = \frac{|1 - z_0\bar{z}|}{|1 - \bar{z}_0 z|} = 1$$

since complex conjugate numbers have the same absolute values, and the interior to interior, since the interior point $z = z_0$ is mapped to $w = 0$.

Conversely, the function $w = (az + b)/(cz + d)$ is non-constant (and invertible) provided that the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-zero. Moreover, the composition of two fractional-linear transformations is again fractional-linear, and is given by the matrix product of their coefficient matrices. The value $w = 0$ corresponds to $z = -b/a =: z_0$, and in particular $a \neq 0$. Pre-composing this map with the inverse of $z \mapsto (z - z_0)/(1 - \bar{z}_0 z)$, we obtain a fractional-linear transformation mapping the unit disk bijectively onto itself and preserving the center of the disk. Explicitly, it has the form $w = z/\mu z + \nu$, and if $|w| = 1$ (and hence $|z| = 1$), then $|\mu z + \nu| = 1$, i.e. $z \mapsto \mu z + \nu$ (the composition of stretch, rotation, and translation) maps the unit circle to itself. This is possible only if both the stretch and translation are trivial, i.e. $|\mu| = 1$ and $\nu = 0$. Thus, the composite map is $z \mapsto e^{i\theta} z$.

3. Let the two fixed points of f be z_0 and z_1. Consider the map $g = h \circ f \circ h^{-1}$, where h is a fractional-linear automorphism of the unit disk mapping z_0 to 0. Then g has two fixed points $w = 0$ and one more, $w = w_0 \neq 0$. Therefore $|g(w_0)| = |w_0|$ for an interior point of the disk, and so by Schwarz' lemma $g(w) = \lambda w$ (with $|\lambda| = 1$), where actually $\lambda = 1$ (since $\lambda w_0 = w_0$. Thus, g is the identity map, and hence $f = h^{-1} \circ g \circ h$ is the identity map too.

4. Let U be a bounded connected component of $\mathbb{C} - |f|^{-1}(c)$ (i.e. $|f| = c$ on the compact boundary $\bar{U} - U$ of the open set U) where f has no zeroes. Then both f and $1/f$ are holomorphic in U and hence obey the maximum modulus principle, i.e. $|f| \leq c$ and $1/|f| \leq 1/c$ in U. This is possible only $|f| = c$ in $\bar{U}$, in which case (according to the maxim modulus principle) f must be constant in U, and hence (by the principle of analytical continuation) everywhere in $\mathbb{C}$.

HW6

1. The exterior $|w| > 1/\epsilon$ of any radius $1/epsilon$ disk contains (infinitely many) strips $2\pi n < \operatorname{Im} w \leq 2\pi(n+1)$ where the $2\pi i$-periodic function e^w assumes all its values $\mathbb{C} - 0$. Therefore the function $e^{1/z}$ in a punctured neighborhood $0 < |z| < \epsilon$ of $z = 0$ assumes all non-zero values infinitely many times, and this is true for any $\epsilon > 0$ no matter how small.

2. $z/\sin z = 1/(1 - z^2/6 + z^4/120 + \cdots$. Consequently

$$\left(\frac{z}{\sin z}\right)^5 = \left[1 + \frac{z^2}{6} - \frac{z^4}{120} + \left(\frac{z^2}{6}\right)^2 + \cdots\right]^5 = \left[1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots\right]^5$$
$$= 1 + 5\frac{z^2}{6} + 5\frac{7z^4}{360} + 10\left(\frac{z^2}{6}\right)^2 + \cdots = 1 + \frac{5z^2}{6} + \frac{3z^4}{8} + \cdots$$

Thus, $\sin^{-5} z = z^{-5} + 5z^{-3}/6 + 3z^{-1}/8 + \cdots$, and so $\operatorname{Res}_{z=0} \sin^{-5} z\, dz = 3/8$.

3. In the ring $2 < |z+1| < 3$, we have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{(z+1)-3} - \frac{1}{(z+1)-2}$$
$$-\sum_{n\geq 0} \frac{(z+1)^n}{3^{n+1}} - \sum_{n>0} \frac{2^{n-1}}{(z+1)^n}.$$

4. Since the Riemann sphere is compact, a meromorphic function f on it has only finitely many poles (one of which could be $z=\infty$). Put $p(z) = \prod_{i=1}^{k}(z-z_i)^{m_i}$ where z_i are finite poles of f, and m_i are their orders. Then $g(z) := p(z)f(z)$ is entire (i.e. holomorphic in $\mathbb{C}$), and has no essential singularity at $z=\infty$. Being entire, g expands in $\mathbb{C}$ into a power series $\sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} a_n/w^n$, where $w = 1/z$. Since neither $f(1/w)$ nor $p(1/w)$ have essential singularities at $w=0$, the same is true about $g(1/w)$, implying that the sum is actually finite. Thus, g is a polynomial, and $f = g/p$ is rational.

HW7

III.20(i). This is a type 2 integral. With $a, b > 0$, we find (using the change $x = \sqrt{a}y/\sqrt{b}$):

$$\int_0^\infty \frac{dx}{(a+bx^2)^n} = \frac{\sqrt{a}}{2a^n\sqrt{b}}\int_{-\infty}^{\infty}\frac{dy}{(1+y^2)^n} = \frac{2\pi i\sqrt{a}}{2a^n\sqrt{b}}\operatorname{Res}_{z=i}\frac{dz}{(z-i)^n(z+i)^n}.$$

The residue at the nth order pole is computed as the value at $z=i$ of

$$\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\frac{1}{(z+i)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{(2n-2)!}{(n-1)!}\frac{1}{(z+i)^{2n-1}}.$$

Therefore the value of the integral is

$$\frac{2\pi i\sqrt{a}}{2a^n\sqrt{b}} \times \frac{(2n-2)!}{2^{2n-1}i[(n-1)!]^2} = \frac{\binom{2n-2}{n-1}\,\pi}{(4a)^{n-\frac{1}{2}}\,b^{\frac{1}{2}}}.$$

III.20(ii). This reduces to type 3b integrals. Assuming $a, b > 0$, we have

$$\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos 2ax - \cos 2bx}{x^2}\,dx$$
$$= \frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{2iax}-e^{2ibx}}{x^2}\,dx + \frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{-2iax}-e^{-2ibx}}{x^2}\,dx$$
$$= \frac{\pi i}{4}\operatorname{Res}_{z=0}\frac{e^{2iaz}-e^{2ibz}}{z^2}\,dz - \frac{\pi i}{4}\operatorname{Res}_{z=0}\frac{e^{-2iaz}-e^{-2ibz}}{z^2}\,dz$$
$$= \frac{\pi i}{4}(2ia - 2ib) - \frac{\pi i}{4}(-2ia + 2ib) = \pi(b-a)$$

Note that near $x=0$, the numerator $\cos 2ax - \cos 2bx = 2(a-b)x^2 + \cdots$ has a 2nd order zero, and so the initial integrand is non-singular at $x=0$. The initial integral converges at $|x| \to \infty$ because the numerator is bounded, and $\int_1^\infty x^{-2}dx = 1 < \infty$. However, the complex integrands $(e^{\pm 2iaz} - e^{\pm 2ibz})/z^2$

have a *first* order pole of at $z = 0$. They have to be integrated along a semicircular contour avoiding $z = 0$ in the upper half-plane in the case of the sign "+", and in the lower half-plane in the case of the sign "−". Respectively, they contribute $\pm\pi i \operatorname{Res}_{z=0}[\cdots]$. The residues *per se* are equal to the derivative of the numerator $e^{\pm 2iaz} - e^{\pm 2ibz}$ evaluated at $z = 0$.

III.20(iii). This is also type 3b integral. The integrand has no singularity at $x = 0$ (since $(\sin x)/x$ is smooth), and converges (due to the sign alternation of $\sin x$) as $x \to \infty$, though not absolutely (since $\int_1^\infty dx/x = \infty$). Using integration over semi-circular contours in the upper half-plane avoiding $z = 0$ (and assuming $a > 0$), we find:

$$\begin{aligned}\int_0^\infty \frac{x^2-a^2}{x^2+a^2}\frac{\sin x}{x}\,dx &= \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty e^{ix}\,\frac{x^2-a^2}{x^2+a^2}\frac{dx}{x}\\ &= \operatorname{Im}\ 2\pi i \operatorname{Res}_{z=ia}\frac{e^{iz}(z^2-a^2)\,dz}{2z(z^2+a^2)} + \operatorname{Im}\pi i \operatorname{Res}_{z=0}\frac{e^{iz}(z^2-a^2)\,dz}{2z(z^2+a^2)}\\ &= \operatorname{Im} 2\pi i\frac{e^{-a}(-2a^2)}{(2ia)^2} + \operatorname{Im}\pi i\frac{-a^2}{2a^2} = \pi\left(e^{-a}-\frac{1}{2}\right).\end{aligned}$$

III.20(iv). Assuming $-1 < a < 1$, we reduce it to type 1 integrals:

$$\begin{aligned}\int_0^\pi \frac{\cos nt\,dt}{1-2a\cos t+a^2} &= \frac{1}{2}\int_{-\pi}^\pi \frac{(\cos nt + i\sin nt)\,dt}{1-2a\cos t+a^2}\\ &= \frac{1}{2}\int_{|z|=1}\frac{z^n}{1-a(z+1/z)+a^2}\frac{dz}{iz} = \pi i \operatorname{Res}_{z=a}\frac{z^n\,dz}{-ia(z-a)(z-1/a)}\\ &= \frac{\pi a^n}{1-a^2}.\end{aligned}$$

The case $|a| > 1$ reduces to the previous one by the change $a \mapsto 1/a$, which yields the answer $\pi a^{-n}/(a^2-1)$.

HW8

1 (III.21). The integral of $dz/(z^2+a^2)\log z$ over the boundary of the annulus $0 < \epsilon \le |z| \le r$ cut along the negative ray of the real axis equals

$$\begin{aligned}&-\int_r^\epsilon \frac{dx}{(x^2+a^2)(\log x+\pi i)} + \int_\epsilon^r \frac{dx}{(x^2+a^2)(\log x-\pi i)}\\ &\qquad -\oint_{|z|=\epsilon}\frac{dz}{(z^2+a^2)\log z} + \oint_{|z|=r}\frac{dz}{(z^2+a^2)\log z}.\end{aligned}$$

The latter two integrals tend to 0 as $\epsilon \to 0$ and $r \to \infty$ respectively (the 1st one, because $2\pi\epsilon\log\epsilon \to 0$, the 2nd one because $2\pi r/r^2\log r \to 0$). The sum of the former two integrals tends to

$$-2\pi i\int_0^\infty \frac{dx}{(x^2+a^2)(\log^2 x+\pi^2)},$$

which is therefore, by the Residue theorem, equal to $2\pi i$ times the sum of residues at the three poles: $z = \pm ai = e^{\log a \pm \pi i/2}$, and $z = 1$ (where the logarithm vanishes). The latter residue is equal to $1/(1+a^2)$ (since $d(\log z)/dz|_{z=1} = 1$). The other two residues yield

$$\sum_{\pm} \mathrm{Res}_{z=ae^{\pm\pi i/2}} \frac{dz}{(z^2+a^2)\log z} = \frac{1}{2ai(\log a + \pi i/2)} + \frac{1}{-2ai(\log a - \pi i/2)}$$
$$= \frac{-\pi i}{2ai(\log^2 a + \pi^2/4)}.$$

Thus,

$$\int_0^\infty \frac{dx}{(x^2+a^2)(\log^2 x + \pi^2)} = \frac{\pi}{2a(\log^2 a + \pi^2/4)} - \frac{1}{1+a^2}.$$

2 (III.22). Due to the parity of the integrand, we have

$$I = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\nu x}\ dx}{e^x + e^{-x} + 2\cosh a} \overset{e^x = z}{=} \mathrm{Re} \int_0^\infty \frac{z^{i\nu}\ dz}{z^2 + 2(\cosh a)z + 1},$$

which seems analogous to both type 3 and type 4 integrals. Let us use the contour of integration as in the case of type 4: the boundary of the annulus between the circles of radii r and $1/r$ centered at $z=0$, cut along the positive part of the real axis. On the circles $z = r^{\pm}e^{i\theta}$, we find $z^{i\nu} = e^{\pm i\nu \log r} - \nu\theta$ bounded uniformly for all $r > 0$. In the limit $r \to \infty$, the integrals over the outer circle vanishes because the rest of the integrand decays as $1/r$, and over the inner circle, it vanishes because the circle's length $2\pi/r$ tends to 0. The integrands on the north and south shores of the cut differ by the factor $e^{-2\pi\nu}$ due to the analytical continuation of $z^{i\nu} := e^{i\nu \log z}$ around the origin. Therefore in the limit $r \to \infty$ we obtain the identity

$$(1 - e^{-2\pi\nu})I = \mathrm{Re}\, 2\pi i \sum_{\pm} \mathrm{Res}_{z_\pm} \frac{z^{i\nu}\ dz}{z^2 + 2(\cosh a)z + 1},$$

where $z_\pm = -e^{\pm a}$ are the roots of the denominator. The residue sum is

$$\frac{e^{i\nu a}e^{-\pi\nu}}{e^{-a} - e^a} + \frac{e^{-i\nu a}e^{-\pi\nu}}{e^a - e^{-a}} = -ie^{-\pi\nu}\frac{\sin \nu a}{\sinh a}.$$

Thus

$$I = \frac{2\pi e^{-\pi\nu}}{1 - e^{-2\pi\nu}} \frac{\sin \nu a}{\sinh a} = \frac{\pi \sin \nu a}{\sinh \nu a\ \sin a}.$$

Note that on the plane of $x = \log z$, the contour of integration we used turns into the boundary of the rectangle with the vertices $\pm \log r, \pm \log r + 2\pi i$, which is exactly the contour suggested in the formulation of the exercise. Thus, our approach, based on the change of variables $x = \log z$ is not really different from the one in the textbook.

3. Since the plane $\mathbb{R}^2$ is simply connected, a function harmonic in the entire plane is the real part of an entire function $f : \mathbb{C} \to \mathbb{C}$. If the given harmonic function $\mathrm{Re}\, f$ is everywhere positive, then $|e^{-f}| = e^{-\mathrm{Re}\, f} < 1$. Thus, by Liouville's theorem, e^{-f} is constant, and hence so is f and its real part.

4. Instead of checking that $g(x,y) := \sin x \cos x/(\cos^2 x + \sinh^2 y)$ satisfies the Laplace equation, it suffices to show that $g(x,y)$ is the real part of the holomorphic function $\tan(x+iy)$ (as the textbook suggests). We have:

$$\tan(x+iy) = -i\frac{e^{i(x+iy)} - e^{-i(x+iy)}}{e^{i(x+iy)} + e^{-i(x+iy)}}$$

$$= -i\frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)} = \frac{\sin x\cosh y + i\cos x\sinh y}{\cos x\cosh y - i\sin x\sinh y}$$

$$= \frac{(\sin x\cosh y + i\cos x\sinh y)(\cos x\cosh y + i\sin x\sinh y)}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y}$$

$$= \frac{\sin x\cos x + i\sinh y\cosh y}{\cos^2 x\cosh^2 y + (1-\cos^2 x)\sinh^2 y} = \frac{\sin x\cos x + i\sinh y\cosh y}{\cos^2 x + \sinh^2 y}.$$

Thus, $\operatorname{Re}\tan(x+iy) = g(x,y)$ indeed.

HW9

1 (V.1). For $|z| \le r < 1$, $|f(z)| \le A|z|$ where $A = \max_{|z|\le r}|f(z)/z|$ (which exists because $f(0)=0$, and hence $f(z)/z$ is holomorphic in $|z|<1$, and hence continuous on the compact disk $|z|\le r$). Consequently $|f(z^n)| \le A|z^n| \le Ar^n$ for $|z|\le r<1$, and therefore $\sum_{n>0}|f(z^n)| \le A\sum_{n>0} r^n < 1$, i.e. the series converges normally on this disk. Since these disks exhaust the open unit disk, the series converges normally (and hence uniformly) on every compact subset of the open unit disk.

2 (V.2). Since $A := \min_{z\in\partial K}|f(z)| > 0$, and $\max_{z\in\partial K}|f_n(z) - f(z)| \le A/2$ for n large enough, we find for any such n that $|(1-\epsilon)f(z) + \epsilon f_n(z)| \ge A/2 > 0$ for all $0\le\epsilon\le 1$ and $z\in\partial K$. Therefore $\oint_{\partial K} d\log f_n = \oint_{\partial K} d\log f$, i.e. the numbers of zeroes inside K for f_n and f coincide.

3. (a) For every point $z_0 \in D$, find a disk $|z-z_0| < 1/n$ still contained in D (it exists since D is open), then pick a point z_0' with rational coordinates inside such that $|z_0' - z_0| < 1/2n$, and associate to z_0 the disk centered at z_0' of rational radius $1/2n$ (whose closure is still contained in D). Given a compact subset $K \in D$, cover it by open disks with rational centers and radii associated as above to the points $z_0 \in K$ (one such disk per each point). Pick a finite subcover (existing due to the compactness of K). Then the closures of these finitely many disks still contained in D and cover K.

(b) The series $\sum_{n>0}(-1)^n/n$ converges (to $-\log 2$), but not absolutely (since $\sum_{n>0} 1/n = +\infty$). Considered as a series of (constant!) holomorphic functions, the series converges uniformly, but not normally.

4 (V.4). The L.H.S. has 1-st order poles at $z = n \mp ai$, $n\in\mathbb{Z}$, with the residues

$$\frac{\pi i\sinh 2\pi a}{\pi(-1)^n\sin\pi(n\mp 2ai)} = \frac{\sinh 2\pi a}{\pm i\sin 2\pi ai} = \mp 1.$$

The same is true about the fractions $\mp 1/(z-n\pm ai)$ on the R.H.S. Analogously to the estimates in no. 2 of V.2 in the textbook (or in the lectures) the R.H.S. converges uniformly on compact subsets to a function meromorphic in $\mathbb{C}$, and both the L.H.S. and R.H.S. are 2-periodic and tend to 0 uniformly with respect to $\operatorname{Re} z$ as $|\operatorname{Im} z| \to \infty$. This implies that *L.H.S.* − *R.H.S.* is an entire functions which tends to 0 at infinity, and hence is identically zero by Liouville's theorem. Using the trigonometric identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ on the L.H.S., and clearing the denominators on the R.H.S., we arrive at the required identity

$$\frac{\pi}{a}\frac{\sinh 2\pi a}{\cosh 2\pi a - \cos 2\pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^n + a^2}.$$

HW10

1. A doubly-periodic meromorphic function f on the elliptic curve $E = \mathbb{C}/\Omega$ with a single pole of order 1 doesn't exist because then the differential form $f(z)dz$ would have a non-zero residue at this pole, and hence the overall non-zero sum of residues — in conflict with the Residue Theorem (according to which $2\pi i \sum \operatorname{Res} f(z)dz = \int_{\partial E} f(z)dz = \int_{\emptyset} f(z)dz = 0$).

2. The Newton equation $\ddot{x} = x + /2x^2$ implies the energy conservation law $\dot{x}^2/2 - (x^2+x^3)/2 = const$ where $const = 0$ due to the initial conditions $x(0) = -1, \dot{x}(0) = 0$. Parameterize the curve $y^2 = x^2 + x^3$ rationally by taking $y = kx$, and hence $x = k^2 - 1$ and $y = k^3 - k$. To solve $dx/dt = \sqrt{x^2+x^3}$, rewrite it as $\int dt = \int y^{-1}dx$ and use the above parameterization to compute the integral on the right:

$$t = \int \frac{d(k^2-1)}{k^3-k^2} = \int \frac{2dk}{k^2-1} = \int \frac{dk}{k-1} - \int \frac{dk}{k+1} = \ln\frac{1-k}{1+k} + Const.$$

From the initial conditions we find that $k = 0$ and hence $Const = 0$. Therefore $e^t = (1-k)/(1+k)$, or $k = (1-e^t)/(1+e^t) = -(\sinh t/2)/(\cosh t/2)$. Thus, $x(t) = k^2 - 1 = -1/\cosh^2(t/2)$.

3. The elliptic curve $(\mathbb{C}-\Omega)/\Omega$ corresponding to a period lattice $\Omega = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}$ is mapped by $\mathbb{C} - \Omega \ni z \mapsto (x,y) = (\wp(z), \wp'(z)) \in \mathbb{C}^2$ bijectively onto the cubic curve in $\mathbb{C}^2$ given by the relation $y^2 - 4x^3 + 20a_2x + 28a_4 = 0$, where

$$a_2 = 3\sum_{\omega\in\Omega-\{0\}} \omega^{-4}, \quad a_4 = 5\sum_{\omega\in\Omega-\{0\}} \omega^{-6}.$$

Replacing each $\omega \in \Omega$ with $k\omega$, where $k \in \mathbb{C} - \{0\}$, results in a new cubic equation where a_2 and a_4 are replaced with a_2k^{-4} and a_4k^{-6} respectively. However, rescaling (x,y) into $(k^{-2}x, k^{-3}y)$ restores the original cubic equation (up to the overall non-zero factor k^{-6} which does not affect the zero locus of the polynomial).

`Remark.` Note that $z \mapsto kz$ identifies $\mathbb{C}/\Omega$ with $\mathbb{C}/k\Omega$, which explains the "restoring" transformation. Namely, it follows from the definition of the Weierstrass $\wp$-function that $\wp_{k\Omega}(kz) = k^{-2}\wp_{\Omega}(z)$ (where the subscript indicates the lattice from which the $\wp$-function is constructed), and respectively $\wp'_{k\Omega}(kz) = k^{-3}\wp'_{\Omega}(z)$.

4. Square lattices $\Omega \subset \mathbb{C}$ are characterized by their invariance under multiplication by i: $i\Omega = \Omega$. Taking in the previous solution $k = i$, we find the transformed cubic equation to be $y^2 - 4x^3 + 20a_2x - 28a_4 = 0$. Therefore the cubic curves corresponding to square lattices (being invariant under the transformation) must have $a_4 = 0$. By suitably rescaling x and y, all such equations can be normalized to $y^2 = x^3 - x$. Likewise, "hexagonal" lattices are those invariant under the multiplication by $e^{\pi i/3}$. The transformed cubic equation becomes $y^2 - 4x^3 - 20e^{-\pi i/3}a_2x + 28a_4 = 0$, and remains unchanged if an only if $a_2 = 0$. All such equations can be rescaled into $y^2 = x^3 - 1$.

Conversely (and more to the point), taking $k = e^{\pi i/2} = i$ in the above Remark, we find that the transformation $(x, y) \mapsto (k^{-2}x, k^{-3}y) = (-x, iy)$ is a symmetry of the curve $y^2 = x^3 - x$ induced by the simultaneous transformation $z \mapsto iz$ of the $\wp$-function's domain and the period lattice — which therefore must be square. Analogously, taking $k = e^{\pi i/3}$, we find that the transformation $(x, y) \mapsto (e^{-2\pi/3}x, -y)$ is a symmetry of the equation $y^2 = x^3 - 1$, implying that $e^{\pi i/3}\Omega = \Omega$.

HW11

1 (V.6). We have:

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Therefore, for a positive integer p,

$$\frac{d}{dpz}\left(\frac{\Gamma'(pz)}{\Gamma(pz)}\right) = \frac{1}{p^2}\sum_{n=0}^{\infty} \frac{1}{(z+n/p)^2} = \frac{1}{p^2}\sum_{r=0}^{p-1} \frac{d}{dz}\left(\frac{\Gamma'(z+r/p)}{\Gamma(z+r/p)}\right).$$

From this, by two integrations, we obtain $\prod_{r=0}^{p-1}\Gamma\left(z + \frac{r}{p}\right) = e^{az+b}\Gamma(pz)$, where a and b are integration constants. To find the constants, substitute $z = 1/p$ and $z = 1$. Since $\Gamma(p) = (p-1)!$, $\Gamma(1) = 1$, and $\Gamma(1 + r/p) = (r/p)\Gamma(r/p)$, we obtain:

$$e^{a/p+b} = \prod_{r=1}^{p-1}\Gamma\left(\frac{r}{p}\right) \quad \text{and} \quad (p-1)!\ e^{a+b} = \frac{(p-1)!}{p^{p-1}}\prod_{r=1}^{p-1}\Gamma\left(\frac{r}{p}\right).$$

Since $\prod_{r=1}^{p-1}\Gamma(r/p) = \prod_{r=1}^{p-1}\Gamma((p-r)/p)$ we conclude, using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, $\prod_{r=1}^{p-1}\Gamma(r/p) = \sqrt{\prod_{r=1}^{p-1}\pi/\sin(\pi r/p)}$. To evaluate

$\prod_{r=1}^{p-1} \sin(\pi r/p)$, note the polynomial identity

$$\prod_{r=1}^{p-1}(x - e^{-2\pi ir/p}) = \frac{x^p - 1}{x - 1} = x^p + x^{p-1} + \cdots + x + 1.$$

Therefore (using the value p of this polynomial at $x = 1$)

$$\prod_{r=1}^{p-1} \sin\frac{\pi r}{p} = \prod_{r=1}^{p-1} \frac{e^{\pi ir/p}}{2i}(1 - e^{-2\pi ir/p}) = \frac{e^{\pi ip(p-1)/2p}}{2i^{p-1}} p = \frac{p}{2^{p-1}}$$

since $e^{\pi i(p-1)/2} = i^{p-1}$. Collecting everything together, we find

$$e^{a/p+b} = (2\pi)^{(p-1)/2} p^{-1/2} \quad \text{and} \quad e^{a+b} = (2\pi)^{(p-1)/2} p^{-1/2} p^{-(p-1)}.$$

Consequently (after some manipulation),

$$e^a = p^{-p} \quad \text{and} \quad e^b = (2\pi)^{(p-1)/2} p^{1/2}.$$

Thus,

$$\Gamma(z)\Gamma\left(z + \frac{1}{p}\right) \cdots \Gamma\left(z + \frac{p-1}{p}\right) = p^{-pz+1/2}(2\pi)^{(p-1)/2}\ \Gamma(pz).$$

2. A family of continuous (say, complex-valued) functions on a metric space K is called equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever the distance $d(x, y)$ between $x, y \in K$ is less than δ, $|f(x) - f(y)| < \epsilon$ for all functions f from the family. The negation of this requirement says: There exists $\epsilon > 0$, two sequences of points $x_n, y_n \in K$, and a sequence f_n of functions from the family, such that $d(x_n, y_n) \to 0$ but $|f_n(x_n) - f_n(y_n)| \geq \epsilon$ for all n. When K is compact, the sequence (x_n, y_n) in $K \times K$ (which is also compact) has a subsequence (x_{n_k}, y_{n_k}) converging to some (x^*, y^*) as $k \to \infty$. Since $d(x_{n_k}, y_{n_k}) < 1/n_k \to 0$ as $k \to \infty$, it follows that $x^* = y^*$. We claim that the sequence f_{n_k} has no uniformly convergent subsequence. For, suppose the opposite, i.e. that the subsequence $f_{n_{k_l}}$, $l = 1, 2, ...$ converges uniformly (as $l \to \infty$) to a function $f^* : K \to \mathbb{C}$, which is therefore continuous. By the triangle inequality

$$|f_{n_{k_l}}(x_{n_{k_l}}) - f^*(x^*)| \leq |f_{n_{k_l}}(x_{n_{k_l}}) - f^*(x_{n_{k_l}})| + |f^*(x_{n_{k_l}}) - f^*(x^*)| \to 0$$

as $l \to \infty$, because on the right, the 1st summand tends to 0 due to $f_{n_{k_l}} \to f^*$, and the 2nd tends to 0 due to $x_{n_{k_l}} \to x^*$ (and the continuity of f^*). Since the same is true for $y_{n_{k_l}}$, we have

$$\lim_{l\to\infty} f_{n_{k_l}}(x_{n_{k_l}}) = f^*(x^*) = f^*(y^*) = \lim_{l\to\infty} f_{n_{k_l}}(y_{n_{k_l}})$$

in conflict with $|f_{n_{k_l}}(x_{n_{k_l}}) - f_{n_{k_l}}(y_{n_{k_l}})| \geq \epsilon$ for all $l = 1, 2, ...$

3. The sequence of functions $f_n := \sin nx$ (on the compact space $\mathbb{R}/2\pi\mathbb{Z}$) together with the two sequences of points $x_n := \pi/2n$, $y_n := -\pi/2n$ satisfy the requirements from the solution of Exercise 2, with $\epsilon = 2$.

4. We have: $1/(1-q^k) = 1+q^k+q^{2k}+\cdots+q^{lk}+\cdots$. Multiplying out these geometric power series with $k = 1, 2, 3, \ldots$, we find that the coefficient at q^n is equal to the number of representations of n as the sum $l_1 1+l_2 2+...+l_k k+\cdots$ (where $l_k \geq 0$ and hence all but finitely many $l_k = 0$). This is indeed the number $P(n)$ of partitions of $n = m_1+\cdots+m_r$, $0 < m_1 \leq \cdots \leq m_m$, where l_k is the number of the terms m_i of the partition, which are equal to k. On the other hand, the reciprocal infinite product $\prod_{k=1}^{\infty}(1-q^k)$ converges uniformly on the disks $|q| \leq r < 1$ by the criterion $\sum_{k>0} |-q|^k \leq r/(1-r) < \infty$. Since the factors have zeroes only on the circle $|q| = 1$ (at the roots of unity), the initial product $1/\prod_{k>0}(1-q^k)$ is holomorphic in $|q| < 1$. Therefore its Taylor expansion at $q = 0$ has convergence radius $\rho \geq 1$. Since at $q = 1$ the series obviously diverges, $\rho = 1$.

Note that this fact imposes some constraints (e.g. in the form of Cauchy's inequalities) on the rate of growth of the sequence $P(n)$.

HW12

1. As it was explained at the end of Lecture 33, the cross-ratios of the permutted 4-tuple of distinct points $(0, 1, \infty, \lambda)$ are obtained by atternating application two transpositions ($(0, \infty)$ and $(0, 1)$):

$$\lambda \mapsto \frac{1}{\lambda} \mapsto \frac{\lambda-1}{\lambda} \mapsto \frac{\lambda}{\lambda-1} \mapsto \frac{1}{1-\lambda} \mapsto 1-\lambda (\mapsto \lambda).$$

The grooup $S_3 = S_4/K_4$ consists of 3 conjugated transpositions, two inverse elements of order 3, and the identity. The fixed points of the transpositions are found from $\lambda = \lambda^{-1}$, $\lambda = \lambda/(\lambda-1)$ and $\lambda = 1-\lambda$, which yields $\lambda = -1, 2$, and $1/2$ respectively (while the 2nd solution in each case, i.e. $\lambda = 1, 0$, and ∞ is a "forbidden" value for configurations of *distinct* points). The fixed points of the order 3 elements are found from $\lambda = (\lambda-1)/\lambda$ or $\lambda = 1/(1-\lambda)$, which both yielding the non-trivial cubic roots of -1: $\lambda = e^{\pm\pi i/3}$. This implies, that up to Möbius transformations and permutations, there is a unique configuration with a symmetry of order 2, and a unique configuration with a symmetry of order 3. The model examples are: $0, \infty, 1, -1$ for the former, and 0 and the 3 cubic roots of unity for the latter.

2. The "Zhukovsky's map" $w = (z + z^{-1})/2$ maps the semi-disk $|z| < 1, Imz < 0$ bijectively onto the upper half-plane $\mathrm{Re}\, w > 0$, and thus $z \mapsto (-iz + iz^{-1})/2$ does the job.

3. Transformations $z \mapsto e^{2\pi ik/n}z$, $k \in \mathbb{Z}/n\mathbb{Z}$, form the cyclic group of order n fixing $z = 0$ and infinity. Therefore, conjugating these transformatiosn by an automorphism h of $\mathbb{C}P^1$ mapping 1 and -1 into 0 and ∞ will do the job. For h, we can take $w = (z-1)/(z+1)$, whose inverse is $z = (1+w)/(1-w)$. Hence the transformations

$$z \mapsto h^{-1}\left(e^{2\pi ik/n}h(z)\right) = \frac{1+e^{2\pi ik/n}\frac{z-1}{z+1}}{1-e^{2\pi ik/n}\frac{z-1}{z+1}} = \frac{z\cos\pi k/n - i\sin\pi k/n}{\cos\pi k/n - iz\sin\pi k/n}$$

form the required subgroup.

4. A holomorphic function f from the punctured disk $0 < |z| < 1$ to the annulus $1 < |w| < 2$ is bounded, and hence doesn't have an isolated singularity at $z = 0$, i.e. it extends holomorphically to the whole disk $|z| < 1$. Then $f(0)$ must lie in the closure of the annulus, i.e. in $1 \leq |w| \leq 2$, and yet be an interior point of it (since a non-constant holomorphic map is open). Assuming f bijective onto the annulus, we would conclude that there exist another point z_0 in the disk with $f(z_0) = f(0)$. But then f could notbe bijective, since the the values w close to $f(0)$ would have at least two inverse images: one in a punctured neighborhood of 0, another in a punctured neighborhood of z_0.

HW13

1. The Möbius transformation $w = (z+1)/(z-1)$ sends the imaginary axis $z = iy$ to the circle $|w| = 1$ (since $|(iy+1)/(iy-1)| = 1$), and the left half-plane to the interior of the disk (since $z = -1$ is mapped to 0). Thus $z \mapsto 1 + 2(z+1)/(z-1) = (3z+1)/(z-1)$ does the job.

2 (VI.2). The line of the centers of two circles in the Euclidean plane intersects them at the 90-degree angles at 4 points, which therefore determine the whole configuration of the two circles. Without loss of generality we may assume that for our configuration (of one circle inside the other), the line of the centers is the real axis. It intersects the circles at 4 points with a certain real value of their cross-ratio, which — we claim — can be realized in the case when the circles are concentric. The real Möbius transformation, mapping the first configuration of the 4 points on the real axis to the second one, preserves the real axis, maps circles to circles, and preserves the angles they make with the axis, and thus transforms the original configuration into the concentric one.

More specifically, we may assume that the outer circle intersects the real axis at $1, -1$, and the inner one at $c + \rho, c - \rho$, where ρ $(0 < \rho < 1)$ is the radius and c the coordinate of the center $(|c| < 1 - \rho)$. The configuration $(z_1, z_2, z, z_4) = (-1, c - \rho, c + \rho, 1)$ has the cross-ratio

$$\frac{(z_2 - z_3)(z_4 - z_1)}{(z_2 - z_1)(z_4 - z_3)} = \frac{(-2\rho)(2)}{(1 - \rho + c)(1 - \rho - c)} = -\frac{4\rho}{(1-\rho)^2 - c^2} < 0.$$

When $c = 0$ and $\rho = r$ varies between 0 and 1, the function $-4r/(1-r)^2$ assumes all negative values, confirming our claim.

3. The map $w = \exp(\pi i z)$ is 2-periodic and maps the lines $z = 0, 1$ to the rays $\arg w = 0, \pi$ respectively, and the strip $0 < \operatorname{Re} z < 1$ to the upper half-plane $0 < \arg w < \pi$. Thus $z \mapsto -i \exp(\pi i z)$ does the job.

4. In homogeneous coordinates $(X : Y : Z)$ on $\mathbb{C}P^2$, the curve $y^2 = 4x^3 - 20a_2x - 28a_4$ is given by the homogeneous cubical equation $Y^2Z = 4X^3 - 20a_2XZ^2 - 28a_4Z^3$, where $x = X/Z$ and $y = Y/Z$. The infinite line of our $\mathbb{C}P^2$ consists of points with $Z = 0$, and intersects our cubical curve at one (hence inflection) point where $4X^3 = 0$, i.e. $X = 0$, while $Y \neq 0$. In the chart $\tilde{x} = X/Y = x/y$, $\tilde{z} = Z/Y = 1/y$ on $\mathbb{C}P^2$ this point is the origin, and our curve is given by the inhomogeneous equation

$$\tilde{z} = 4\tilde{x}^3 - 20a_2\tilde{x}\tilde{z}^2 - 28a_4\tilde{z}^3.$$

This is an implicit equation on $\tilde{z}$ as a function of $\tilde{x}$ near $\tilde{x} = 0$. In the power series form, the solution[1] has a triple zero at $\tilde{x} = 0$: $\tilde{z} = 4\tilde{x}^3 +$ (higher order terms). (Indeed, $\tilde{z} = 0$ is the tangent line at an inflection point!) We are ready now to examine the differential 1-form $(dx)/y$ near the point at infinity. We have: $y = 1/\tilde{z}$, $x = \tilde{x}/\tilde{z}$, and

$$\frac{dx}{y} = \frac{d(\tilde{x}/\tilde{z})}{1/\tilde{z}} = d\tilde{x} - \tilde{x}\frac{d\tilde{z}}{\tilde{z}}.$$

In this expression, the logarithmic differential $(d\tilde{z})/\tilde{z}$, when restricted to our curve, i.e. expressed in terms of the local coordinate $\tilde{x}$ near $\tilde{x} = 0$, has a 1-st order pole (as any logarithmic differential) with the residue 3 (equal to the order of the zero of $\tilde{z}$ as a function of $\tilde{x}$). Thus, $(dx)/y = (-2 + \text{higher order terms})\, d\tilde{x}$, i.e. has no pole at infinity, and doesn't vanish there either.

[1] In fact, the power series expansion for this function can be found by the method of iterations. Starting from $\tilde{z} = 0$, we obtain the next approximation of $\tilde{z}$ (on the left) by substituting the previous approximation on the right: First $\tilde{z} = 4\tilde{x}^3$, next $\tilde{z} = 4\tilde{x}^3 - 20a_4\tilde{x}(4\tilde{x}^3)^2 - 28a_4(\tilde{x}^3)^3$, and so on.