Math 140. Solutions to homework problems.

Homework 1. Due by Tuesday, 01.25.05

1. Let D_d be the family of domains in the Euclidean plane bounded by the smooth curves ∂D_d equidistant to a bounded convex domain D_0. How does the perimeter $Length(\partial D_d)$ depend on the distance d between ∂D_d and D_0?

Solution 1. Use the result from class: $Area(D_d) = Area(D_0) + d \cdot Length(\partial D_0) + \pi d^2$. This implies $Length(\partial D_0) = d\, Area(D_d)/d(d)\|_{d=0}$. Since any of the domains D_d can be taken on the role of D_0, we find $Length(\partial D_d) = d\ Area(D_d)/d(d) = Length(\partial D_0) + 2\pi d$.

Solution 2. Use the method from class. For convex polygons, $Length(\partial D_d) = Length(\partial D_0) + 2\pi d$ by direct observation. We obtain the same result for arbitrary convex domains D_0 by approximating them with polygons and passing to the limit.

Solution 3. Avoid limits and polygons. Let $s \mapsto r(s)$ be the counter-clockwise arc-length parameterization of ∂D_0. Then ∂D_d can be parameterized by adding to $r(s)$ the d-multiple of the (unit) vector dr/ds rotated 90 degrees clockwise (we denote the rotation by J: $s \mapsto f_d(s) := r(s) + d \cdot Jdr(s)/ds$. The velocity vector $df_d(s)/ds = dr/ds + d \cdot Jd^2r/ds^2 = (1 + d \cdot k(s))dr(s)/ds$ because the acceleration $d^2r(s)/ds^2$ of the original curve is proportional to $-Jdr(s)/ds$ with the proportionality coefficient $k(s)$ (by the very definition of curvature k). Since $|dr/ds| = 1$ and $k > 0$ (convexity), we have

$$Length(\partial D_d) = \oint |df_d(s)|ds = \oint ds + d \cdot \oint k(s)ds = Length(\partial D_0) + 2\pi d.$$

2. Verify the invariance of the arc length $\int_a^b \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\ dt$ under reparameterizations $t = t(\tau)$.

By the chain rule, and the variable change law in integrals, we have

$$\int_\alpha^\beta \sqrt{(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2}\ d\tau = \int_\alpha^\beta \sqrt{\dot{x}^2(\frac{dt}{d\tau})^2 + \dot{y}^2(\frac{dt}{d\tau})^2}\ d\tau$$

$$= \int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{y}^2}\ |\frac{dt}{d\tau}|\ d\tau = \int_{t(\alpha)}^{t(\beta)} \sqrt{\dot{x}^2 + \dot{y}^2}\ dt.$$

3. (a) Prove the formula $k = (\ddot{x}\dot{y} - \ddot{y}\dot{x})/(\dot{x}^2 + \dot{y}^2)^{3/2}$ for the curvature of a regular parameterized plane curve $t \mapsto (x(t), y(t))$.

The determinant $(\ddot{x}\dot{y} - \ddot{y}\dot{x})$ is (up to a sign, which actually should be reversed to agree with our orientation conventions) the area of the parallelogram spanned by the velocity $\dot{r}$ and the acceleration $\ddot{r}$ and thus equals $|\dot{r}| \times |a_n|$ ("base times height"). The curvature $k = |a_n|/|\dot{r}|^2$ is therefore obtained from the area by dividing it by $|\dot{r}|^3 = (\dot{x}^2 + \dot{y}^2)^{3/2}$.

(b) Compute the curvature of the graph of a smooth function $y = f(x)$.

Parameterizing the graph as $t \mapsto (x, y) = ((t, f(t))$, we obtain $k(x) = f''(x)/(1 + f'(x))^{3/2}$.

(c) Take $f = x^a/a$ and find the limit of curvature at $x = 0$ for $a = 5/2, 2, 3/2, 1, 1/2$.

At the origin, the curve $y = x^{5/2}$ has the curvature 0 (since it is best approximated by the parabola $y = kx^2/2$ with $k = 0$); the curve $y = x^2/2$ is the parabola with $k = 1$; the curve $y = x^{3/2}/(3/2)$ has the curvature (according to part (b)) $k(x) = x^{-1/2}/2/(1 + x^{1/2})^{3/2}$ which tends to ∞ as x approaches 0; $y = x$ is a straight line and has $k = 0$; and $y = x^{1/2}/(1/2)$ means $y^2/4 = x$, which is a parabola again with the curvature at the origin equal to $1/2$.

4. Draw the typographic symbol ∞ ("infinity" or "figure eight") increased 100 times and then draw an equidistant curve as follows: orient all normal lines to the large figure eight in a continuous fashion, and connect all points removed 1 *cm* from the large figure eight in the *positive* normal direction. Which curve is longer — the large figure eight or the curve equidistant to it?

Using any of the methods from Problem 1 (e.g. approximating the curve with polygons) one concludes that

$$Length(\partial D_d) = Length(\partial D_0) + 2\pi d \times \text{ (rotation index of the tangent line)}.$$

Since the tangent line to "figure eight" makes 0 number of turnes, the equidistant curve has the same length as the "figure eight".

Homework 2. Due by Tuesday, 02.01.05

1. Show that maps $\mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \mathbf{y}$ which preserve all Euclidean distances are given by linear inhomogeneous functions, namely by compositions of translations with rotations or reflections.

Solution. Any isometry F maps a triangle ABC to another triangle $A'B'C'$ with the same pairwise distances between the vertices. The "side-side-side proposition" in elementary Euclidean geometry guarantees that the triangle ABC can be identified with $A'B'C'$ by a suitable composition G of translation with rotation or reflection. Thus the composition $J = G^{-1} \circ F$ is an isometry fixing the vertices: $J(A) = A, J(B) = B, J(C) = C$. We claim that any isometry J fixing three non-colinear points A, B, C is the identity, and thus $F = G$. To justify the claim, note that a point P and and its image $P' = J(P)$ have the same distance to the fixed points A, B, C of the isometry J. If $P \neq P'$ then all points equidistant to them are situated on the line perpendicular to the segment PP' and bisecting it. Since A, B, C are not on the same line, we have $P = J(P)$ for all points P.

2. Compute the curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $(x_0, y_0) = (0, b)$.

Solution. The best approximation of the ellipse near $(0, b)$ with a parabola of the form $Y = kX^2/2$ can be computed from the esllipse's equation:

$$b-y = b-\sqrt{b^2(1-\frac{x^2}{a^2})} = b-b\sqrt{1-\frac{x^2}{a^2}} = b-b(1-\frac{x^2}{2a^2}+...) = \frac{b}{a^2}\frac{x^2}{2}+...$$

Thus $X = x, Y = b - y$, and the curvature in question is $k = b/a^2$.

3. Let $t \mapsto (x(t), y(t))$ be a *closed* regular plane curve. Let $t \mapsto (\dot{x}(t), \dot{y}(t))$ be the closed regular plane curve formed by the velocity vectors. Prove that the integral

$$\frac{1}{2\pi} \oint \frac{\dot{x}d\dot{y} - \dot{y}d\dot{x}}{\dot{x}^2 + \dot{y}^2}$$

is an integer. Point out geometric interpretations of this integer in terms of the velocity curve and of the original curve.

Solution. Taking $P = -\dot{y}/(\dot{x}^2 + \dot{y}^2)$ and $Q = \dot{x}(\dot{x}^2 + \dot{y}^2)$ (where $\dot{x}, \dot{y}$ are just names of independent variables), we find $Q_{\dot{x}} = P_{\dot{y}}$. Thus, by Green's Theorem, $\int_{\partial D}(Pd\dot{x} + Qd\dot{y}) = 0$ over the boundary ∂D of any region D on the $(\dot{x},\dot{y})$-plane which does not contain the origin. This implies that the integral over a simple closed curve going counter-clockwise around the origin is equal to the similar integral over a small circle centered at the origin. The latter integral can be computed using the parameterization $\dot{x}=\epsilon\cos t,\ \dot{y}=\epsilon\sin t$ and is equal to 2π (regardless of the radius ϵ). Generalizing the concliusion to closed non-simple curves (i.e. those which are allowed to self-intersect), we can partition them into simple parts between self-intersections and arrive at the conclusion that $\oint(Pd\dot{x}+Qd\dot{y})$ is an integer multiple of 2π.

On the other hand, $Q_{\dot{x}}=P_{\dot{y}}$ mean that, at least locally, there is a function with the partial derivatives P and Q, and knowing this it is not hard to guess the function. Namely, differentiating $\theta(\dot{x},\dot{y})=\arctan(\dot{y}/\dot{x})$, we find $d\theta=P\ d\dot{x}+Q\ d\dot{y}$. Therefore the integral $\int d\theta$ computes the increment of the polar angle of the vector $(\dot{x},\dot{y})$. Thus the above integer is interpreted as the total number of turns the velocity curve makes around the origin, or equivalently, as the rotation number the oriented tangent line of the original closed curve $t\mapsto(x(t),y(t))$.

4. Compute the cutvature and torsion of the parameterized space curves (t,t^2,t^3), (t,t^2,t^4), (t,t^3,t^4) at $t=0$.

The curve (t,t^3,t^4) has an inflection point at the origin and thus has at this point curvature $k=0$ and torsion τ undefined.

The other two curves have the osculating plane $z=0$ at the origin and project to this plane to the parabola $y=x^2$ with the curvature $k=2$.

To compute the torsion of the curve $r(t)=(t,t^2,t^3)$, we find its velocity $\dot{r}=(1,2t,3t^2)$, acceleration $\ddot{r}=(0,2,6t)$, and the binormal vector

$$b=\frac{\dot{r}\times\ddot{r}}{\|\dot{r}\times\ddot{r}\|}=\frac{(6t^2,-6t,2)}{\sqrt{4+36t^2+36t^4}}=(0,0,1)+t(0,-3,0)+...$$

Therefore at the origin we have $db/ds=(db/dt)(dt/ds)=(0,-3,0)$ since $dt/ds=\|\dot{r}(0)\|=1$. Thus $db/ds=-3n$, and $\tau=-3$.

A similar computation for the curve (t,t^2,t^4) will inevitably yield $\tau=0$ since near the origin the curve differs from a plane curve only by the 4-th order terms t^4.

Homework 3. Due by Thursday, 02.10.05

1. Prove that a space curve with the identically zero torsion is contained in a plane.

Solution. Let $k(s) > 0$ be the curvature of the space curve as a function of the arc length parameter $s \in (a, b)$. By the fundamental theorem for plane curves there exists a plane curve with this curvature function. Considered as a space curve, this curve has the same curvature function and identically zero torsion. By the fundamental theorem for space curves, this plane curve can be identified with the original space curve by a rigid motion of the space. Thus the original curve is contained in a plane.
bigskip
2. The inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is related to the length $\| \cdot \|$ by means of the *polarization identity*:

$$\langle x, y \rangle = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2).$$

Prove this identity, and deduce from it that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is any length-preserving linear transformation, then T preserves the inner product, i.e.

$$\langle T(x), T(y) \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^n$.

Solution. The polarization identity follows from bilinearity and symmetry properties of the inner product

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$

and expresses the inner product in terms of the vector sum and length. Since $T(x + y) = Tx + Ty$ (by linearity of T), the lengths of $x, y, x + y$ coincide respectively with those of $Tx, Ty, Tx + Ty$, and therefore the inner products $\langle x, y \rangle$ and $\langle Tx, Ty \rangle$ coincide too.

3. Let $A(t)$ be an anti-symmetric $n \times n$-matrix depending continuously on t, and U_0 be an orthogonal $n \times n$-matrix (i.e. $A^* = -A$, and $U_0^* = U_0^{-1}$, where * means transposition).

Consider the system $\dot{M} = A(t)M$ of n^2 linear ordinary differential equations in the space $\mathbb{R}^{n^2}$ of $n \times n$-matrices M.

Prove that the solution $t \mapsto M(t)$ to this system satisfying the initial condition $M(t_0) = U_0$ consists of orthogonal matrices $M(t)$.

Solution. Since $M^*(t_0)M(t_0) = U_0^*U_0 = I$, it suffices to prove that $M^*(t)M(t)$ does not depend on t. Differentiating, we find

$$\frac{d}{dt}M^*M = \dot{M}^*M + M^*\dot{M} = (AM)^*M + M^*(AM) = M^*(A^* + A)M = 0.$$

4. Does there exist a *closed* space curve with constant nonzero curvature and (somewhere) nonzero torsion?

Solution. The answer is "yes". For a regular space curve, having somewhere non-zero torsion is the same as not fitting any plane. To have the constant curvature $k = 1$, a space curve $s \mapsto r(s)$ parameterised by arc length must have unit acceleration $\|d^2r/ds^2\|$, or equivalently, the velocity curve $s \mapsto v(s) = dr/ds$ mast be parameterised by the arc length too. Reformulating the problem in terms of the velocity curve, we are therefore looking for a closed curve on the unit sphere with the center of the sphere being the mass center of the curve with respect to the mass distribution proportional to the arc length, and require that the curve does not fit a plane passing through the origin, i.e. it is different from an equator. To construct such a curve, make a "bump" somewhere on the equator and repeat the bump centrally symmetrically on the opposite side of the equator to guarantee that the mass center is at the origin.

Homework 4. Due by Thursday, 02.17.05

1. For each of the 5 Platonic solids (tetrahedron, cube, octahedron, icosahedron and dodecahedron), compute the *angular defect* at each vertex, i.e. the difference between 2π and the sum of face's angles adjecent to this vertex. What do the angular defects of all vertices add up to?

Solution. Vertices of T, O, I are adjecent to respectively $3, 4, 5$ faces which are regular triangles with the angles $\pi/3$. Thus the vertices of T, O, I have angular defects respectively $2\pi/2, 2\pi/3, 2\pi/6$. When multiplied by the number of vertices $4, 6, 12$, these yield 4π. At each of the 8 vertices the cube has the angular defects $2\pi - 3\pi/2 = \pi/2$ which add up to 4π. The dodecahedron has 20 vertices each adjecent to 3 pentagonal faces with the angles $3\pi/5$. Thus each angular defect is $2\pi - 9\pi/5 = \pi/5$, and the total defect is 4π again.

2. Given two skew-lines in $\mathbb{R}^3$ (i.e. two straight lines which are not parallel and have no common points), rotate one of them about the other, find the equation of the resulting surface of revolution and show that the surface is a hyperboloid of one sheet.

Solution. Let the axis of rotation be the z-axis, and the other line be $z = kx, y = b$. Rotating the points $(x, y, z) = (t, b, kt)$ through the angle ϕ around the z-axis we obtain our surface of revolution parameterized by (t, ϕ):

$$x = t\cos\phi - b\sin\phi, \quad y = t\sin\phi + b\cos\phi, \quad z = kt.$$

To eliminate t and ϕ, we find $x^2 + y^2 = b^2 + t^2 = b^2 + z^2/k^2$ and finally

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{(kb)^2} = 1,$$

which is the standard equation of a one-sheeted hyperboloid of revolution.

Monic degree-3 polynomials $P(x) = x^3 + ax^2 + bx + c$ form a 3-dimensional space with coordinates (a, b, c). In this space, consider the *discriminant* Δ — the surface formed by those polynomilas which have a multiple root. Such polynomials have the form $P(x) = (x-u)^2(x-v)$ which provides a parameterization of Δ by (u, v).

3. (a) Sketch the section of the discriminant by the plane $a = 0$.

(b) Show that the transformation $P(x) \mapsto P(x + t)$ defines a (non-linear) flow in the space of polynomials which preserves Δ and transforms the plane $a = 0$ to $a = 3t$. Use this to sketch Δ.

Solution. The identity $x^3 + ax^2 + bx + c = (x - u)^2(x - v)$ yields the parameterization of Δ:

$$(*) \qquad a = -2u - v, \quad b = u^2 + 2uv, \quad c = -u^2v.$$

When $a = 0$, we have $v = -2u$, and the section of the discriminant becomes the semi-cubical parabola $b = -3u^2, c = 2u^3$ on the (b, c)-plane.

The translation $x \mapsto x + t$ transfroms $(x - u)^2(x - v)$ into $(x - (u - t))^2(x - (v - t))$ with shifted but still multiple roots. Therefore the corresponding flow

$$(a, b, c) \mapsto (P''(t)/2, P'(t), P(t)) = (3t + a, 3t^2 + 2at + b, t^3 + at^2 + bt + c)$$

in the space of polynomials preserves Δ and maps the plane $a = 0$ to $a = 3t$. Thus the sections of Δ by the planes $a = const$ are also semi-cubical parabolas subject to non-linear changes of variables on the (b, c)-plane depending on the *const*. We can conclude that Δ looks like the cartesian product of the semi-cubical parabola and the line but distorted by a non-linear change of coordinates.

4. (a) Show that singular points of Δ form the curve C consisting of polynomials $(x-u)^3$ with a triple root, and show that Δ is the *osculating surface* of C (i.e. is swept by tangent lines to C).

(b) Sketch the osculating surface of the curve (t, t^2, t^3) together with its osculating plane at $t = 0$. (*Hint:* the curve can be identified with C by stretching the coordinates.)

Solution. Computing the Jacobi matrix of the parameterization (*) and equating its 2×2-minors to zero we find

$$(a_u, b_u, c_u) \times (a_v, b_v, c_v) = (-2, 2u + 2v, -2uv) \times (-1, 2u, -u^2) =$$
$$= (2u^2v - 2u^3, 2uv - 2u^2, 2v - 2u) = (0, 0, 0)$$

and conclude that singular points of Δ have $v = u$ and form the curve $(x-u)^3$ with $(a, b, c) = (-3u, 3u^2, -u^3)$. The tangent line to this curve at the point $P = (x-u)^3$ is spanned by the velocity vector $dP/du = -3(x-u)^2$ and can be parameterised by t as

$$(x-u)^3 - 3t(x-u)^2 = (x-u)^2(x-u-3t) = (x-u)^2(x-v) \text{ if } v = u+3t.$$

Thus the union of the tangent lines coincides with Δ.

The answer to 4b is shown on Figure 1.

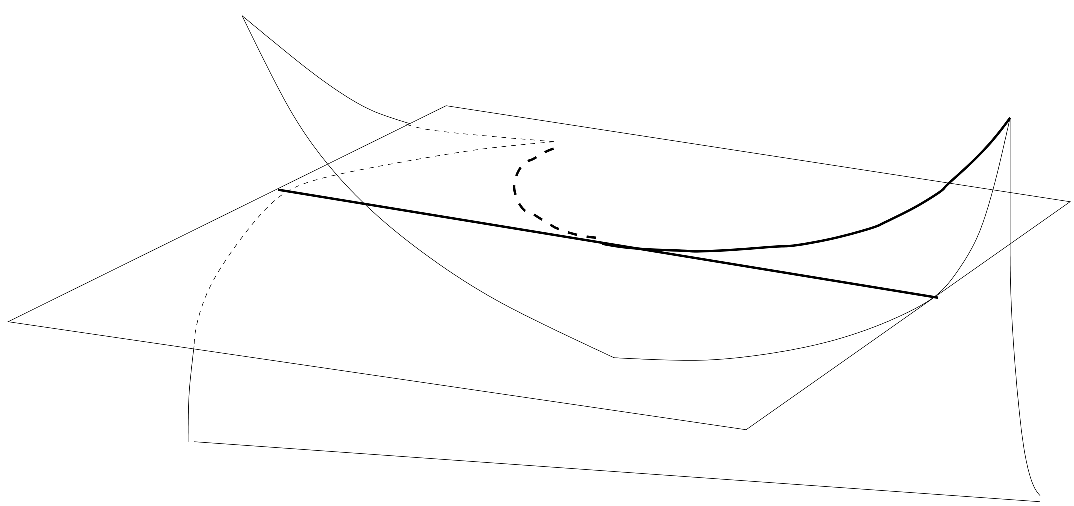

FIGURE 1

Homework 5. Due by Thursday, February 24.

1. Show that the Riemannian area

$$\int\int_D \sqrt{AC-B^2}\; dUdV$$

of a region D on the plane equipped with a Riemannian metric

$$A(U,V)(dU)^2 + 2B(U,V)(dU)(dV) + C(U,V)(dV)^2$$

is invariant with respect to changes of variables $U = U(u,v), V = V(u,v)$.

Solution. Under the change of variables we have

$$dU = U_u du + U_v dv, \quad dV = V_u du + V_v dv,$$

and respectively $AdU^2+2BdUdV+CdV^2 = adu^2+2bdudv+cdv^2$, where $a = AU_u^2+2BU_uV_u+CV_u^2$, $b = AU_uU_v+B(U_vV_u+U_uV_v)+CV_uV_v$, $c = AU_v^2+2BU_vV_v+CV_v^2$. We compute: $ac-b^2 = (AC-B^2)(U_uV_v-U_vV_u)^2$. Let D' denote the region D in the new coordinates. Then

$$\int\int_{D'} \sqrt{ac-b^2}dudv = \int_{D'} \sqrt{AC-B^2}(U(u,v),V(u,v))\, |U_uV_v - U_vV_u|\; dudv$$

$$= \int\int_D \sqrt{AC-B^2}(U,V)\; dUdV$$

due to the rule of change of variables in double integrals.

2. (a) Compute the Riemannian metric, induced by the standard embedding of the sphere of radius r into the Euclidean 3-space, in terms of spherical coordinates.

(b) Using (a) compute the Riemannian area of the spherical triangle bounded by the equator and by two meridians making the angle Φ to each other.

Solution. On the sphere of radius r we have $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\sin\theta$, and

$$dx = -r\sin\theta\cos\phi d\theta - \cos\theta\sin\phi d\phi,\; dy = -r\sin\theta\sin\phi d\theta + \cos\theta\cos\phi d\phi,$$

$dz = r\cos\theta d\theta$. Respectively,

$$dx^2 + dy^2 + dz^2 = r^2 d\theta^2 + r^2\cos^2\theta d\phi^2.$$

The triangle

$$\Delta = \{(\theta,\phi) | 0 \le \theta \le \pi/2,\; 0 \le \phi \le \Phi\}$$

has the area

$$\int\int_\Delta \sqrt{AC-B^2} d\theta d\phi = r^2 \int_0^{\pi/2} \cos\theta d\theta \int_0^{\Phi} d\phi = r^2\Phi.$$

3. In cartography, a popular way to obtain a plane image of the Earth's surface is based on the projecting the sphere $x^2+y^2+z^2=r^2$ onto the cylinder $x^2+y^2=r^2, |z|\leq r$ by rays in the planes $z = const$ radiating away from the z-axis:

$$(x,y,z)\mapsto (x\sqrt{r^2-z^2}, y\sqrt{r^2-z^2}, z).$$

Show that this projection of the sphere to the cylinder preserves areas. Is this projection an isometry, i.e. preserves lengths of all curves?

Solution. Since rotations about the z-axis preserve the sphere, the cyclinder and the projection, the factor by which the projection stretches areas is independent of the sperical θ but could be a function of ϕ. Solution to Problem 2 shows that a narrow annulus bewtween ϕ and $\phi+\Delta\phi$ on the sphere has the area $2\pi r^2\Delta\phi\cos\phi$. Its projection to the cylinder has width $r\Delta\phi\cos\phi$ and length $2\pi r$ and thus has the same area for any ϕ. Thus the area-stretching factor does not depend on ϕ either and is equal to 1.

The map is not an isometry (e.g. parallels are shorter on the sphere than on the cylinder).

4. Compute the geodesic curvature of the circle $z=r/2$ on the surface of the sphere $x^2+y^2+z^2=r^2$ and compare it with the curvature of the same circle considered as a curve in the space.

Solution. Expanding the metric found in Problem 2 in spherical coordinates near the origin $\theta=0,\phi=0$, we find

$$r^2d\phi^2+r^2(1-\theta^2/2+...)^2d\theta^2 = d(r\theta)^2+d(r\phi)^2+ \text{ terms of order } \geq 2\ ,$$

i.e. the coordinate system $(u,v)=(r\theta,r\phi)$ is planar (in fact at every point of the equator $\theta=0$. Rotating the sphere we can transform the section $z=r/2$ into $x+\sqrt{3}z=r$ passing through $\theta=\phi=0$. It has the equation

$$1=\cos\theta\cos\phi+\sqrt{3}\sin\theta=(1-\frac{\theta^2}{2})(1-\frac{\phi^2}{2})+\sqrt{3}\theta+...$$

Neglecting with terms of order >2 we obtain in the coordinate system (u,v)

$$0=2\sqrt{3}ru-v^2-u^2=3r^2-(u-\sqrt{3}r)^2-v^2,$$

which is a circle of radius $\sqrt{3}r$. Thus the geodesic curvature is equal to $1/\sqrt{3}r$. This is half the curvature in the space of the same curve (which is a circle of radius $\sqrt{3}r/2$).

Homework 6. Due by Tuesday, March 8.

1. Perform the parallel transport around the loop on the sphere $x^2+y^2+z^2=r^2$ cut out by the plane $z=r/2$. Find the area enclosed by this curve on the sphere.

Solution. The angle between a transported vector and the direction of the curve along which it is transported (counted *from* the vector *to* the direction) is increased with the rate equal to the geodesic curvature k_g of the curve (because this is true in planar coordinates). Thus the rotation angle ϕ (from the vector to the direction of the curve at the reference point) under the parallel transport around a smooth closed curve C is equal to *minus* the total geodesic curvature of the curve: $\phi = -\int_C k_g dLength$. In Problem of Homework 5 we have found $k_g = 1/\sqrt{3}r$, and the length of the circle equals $\sqrt{3}\pi r$, so that $\phi=-\pi$. Thus as the result of the parralel transport along this curv, all vectors reverse their direction.

The area enclosed by this curve can be computed by pojecting the region to the cylinder as in Problem 3 of Homowork 5 and is found to be the quarter of the total surface area of the sphere, i.e. πr^2. For $r=1$ the answer is π. Is this just a coincidence?

2. Prove that meridians on a surface of revolution are geodesics. Are all paralleles geodesics too?

Solution. It is easy to see that the sign of geodesic curvature (properly defined the same way as on the Euclidean plane) is reversed under the change of orientation of the surface.

The reflection about the plane containing the meridian preserves the surface of revolution and leaves all points of the meridian fixed. Thus the signed geodesic curvature of the meridian at each point satisfies $k_g = -k_g$, i.e. $k_g = 0$.

Moreover, the same is true for any fixed point curve of an isometric reflection on a Rienan surface.

Parallels are typically not geodesics (e.g. non-equatorial parallels on the sphere are not).

3. Express the rotation angle under parallel transport around a curvilinear triangle on a Riemannian surface in terms of angles at the vertices of the triangle and the total geodesic curvature of its sides.

Solution. Reasoning as in Problem 1 about the angle between the transported vector and the direction of sides of the triangle δ we conclude that the transport angle (which is defined modulo 2π anyway) is equal to (s is the arc length parameter on $\partial\Delta$)

$$\phi_\Delta = \alpha + \beta + \gamma - 3\pi - \int_{\partial\Delta} k_g(s)\ ds.$$

Similarly, for a curvilinear n-gon P with the angles $\alpha_1, ..., \alpha_n$ the answer will be

$$\phi_P = \sum \alpha_i - n\pi - \int_{\partial P} k_g(s)\ ds.$$

4. Prove the n-dimensional version of the Key Lemma: *Any Riemannian metric in $\mathbb{R}^n$ near any point is Euclidean modulo terms of order ≥ 2 in a suitable local coordinate system, which is unique up to linear orthogonal transformations and modulo terms of order ≥ 3.* Deduce that Riemannian metrics do not have local invariants depending only on the derivatives of the metric of order ≤ 1.

Solution. Applying the change of variables

$$dU_k = u_k + \sum_{ij} a^k_{ij} u_i u_j + (\text{terms of order} \geq 3),\ \ k = 1, ..., n,$$

(where $a^k_{ij} = a^k_{ji}$) to te Riemannian metric

$$(dU_1)^2 + ... + (dU_n)^2 + (\text{terms of order} \geq 2)\ ,$$

we find

$$\sum (du_k)^2 + 2\sum_i u_i \sum_{jk} a^k_{ij} du_j du_k + \cdots .$$

Thus the null-space of the linear map from the space of n-tuples of quadratic forms in u_i with coeffcieints a^k_{ij} to the space of n-tuples of quadratic forms in du_k (i.e. the n linear terms of the metric) consists of solutions to the system of linear equations

$$a^k_{ij} + a^j_{ik} = 0,\ \ a^k_{ij} = a^k_{ji},\ \ \forall i, j, k = 1, ..., n.$$

Fir each i, j, k these equations imply

$$a^k_{ij} = -a^j_{ik} = -a^j_{ki} = a^i_{kj} = a^i_{jk} = -a^k_{ji} = -a^k_{ij},$$

and therefore $a^k_{ij} = 0$. Since an injective linear map between spaces of the same dimension is an isomorphism, we conclude that any linear terms of a Riemannian metric (expanded near a point by Taylor's formula) can be obtained from a metric which is Euclidean modulo terms of order ≥ 2 by a change of variables $U(u) = u + ...$ with uniquely determined quadratic part.

Homework 7. Due by Tuesday, March 15.

1. Show that the Gaussian curvature $K(p)$ at a point p on a Riemann surface depends only on the derivatives of order ≤ 2 of the Riemannian metric at this point.

Solution. As we have shown in the class, the Gaussian curvature $K(p)$ of a Riemannian metric depends only on the 2nd derivatives of the coefficients of the metric with respect to a coordinate system planar at p. (This is because higher order terms of the metric contribute to the parallel transport angle around a parallelogram of size $\approx \epsilon$ only in the order ϵ^3 and higher.) On the other hand, the coefficients of the metric in a coordinate system planar at p depend on the coefficients of the metric in the original coordinate system and the 1st derivatives of the functions defining the change of variables. According to the Key Lemma, Taylor expansions of these functions up to order 3 inclusively (and therefore Taylor expansions up to order 2 of their 1st derivatives) are determined by the values at p of the 1st (and 0) order derivatives of the coefficients of the metric.

2. Let us call a coordinate system (u, v) on a Riemann surface *Gaussian* with respect to the point $(0, 0)$ if the Riemannian metric has the form

$$(du)^2 + (dv)^2 - \frac{K}{3}(udv - vdu)^2 + \text{ (terms of order} \geq 3).$$

Show that the coefficient K equals the Gaussian curvature of this Riemannian metric at the origin.

Solution. As we have shown in the class, the Gaussian curvaure $K(p)$ depends linearly on the 2nd derivatives at p of the metric in a coordinate system planar w.r.t. p. (This is because the dimension $length^{-2}$ of the Gaussian curvature coincides with the dimension of the 2nd derivatives of the metric's coefficients.) On the other hand, we proved in the class existence of a Gaussian coordinate system for each point p and found in the proof that the local invariant K of the metric is equal to the value at $(u, v) = (0, 0)$ of the linear combination $B_{uv} - A_{vv}/2 - C_{uu}/2$ of the metric's coefficients in a planar coordinate system. In a Gaussian coordinate system, the 2nd derivatives of the metric's coefficients are proportional to K, and thus the Gaussian curvature $K(p)$ has to be proportional to K with the proportionality coefficient independent on metric. Thus to find the coefficient it suffices to consider one example with $K(p) \neq 0$. We take the unit sphere in spherical coordinates to find the metric $(d\theta)^2 + \cos^2\theta(d\phi)^2 = (d\theta)^2 + (d\phi)^2 - \theta^2(d\phi)^2 + \ldots$ planar w.r.t. the point $\theta = 0$, $\phi = 0$, i.e. $B_{uv} = A_{vv} = 0$, and $C_{uu} = -2$. Thus

$K = 1$ which coincides with the Gaussian curvature of the unit sphere. Therefore $K = K(p)$.

3. Show that in a Gaussian coordinate system geodesics passing through the origin coinside with straight lines up to order 3, i.e. not only have $r''(0) = 0$ but also $r'''(0) = 0$.

Hint: check that any straight line through the origin is a symmetry line of the metric $(du)^2 + (dv)^2 - K(udv - vdu)^2/3$ (without higher order terms).

Solutions. The reflection about any line throught the origin preserves $(udv - vdu)^2$, and hence the line is a geodesic of the metric $(du)^2 + (dv)^2 - K(udv - vdu)^2/3$. When terms of order ≥ 3 are in presence, their contribution to the parallel transport angle α along the line from the origin to a point distance ϵ away is estimated as $\alpha \approx Const \int_0^\epsilon x^2 dx = const\epsilon^3$.] This characterizes the direction r' of the geodesic tangent to the line at the origin as deviating only to order ϵ^3 from $r'(0)$, which correponds to $r''(0) = r'''(0) = 0$.

4. Show that any quadratic form $ax^2 + 2bxy + cy^2$ on the plane can be transformed by a linear change of coordinates to one and only one of the following six forms

$$0,\ X^2,\ -Y^2,\ -X^2 - Y^2,\ X^2 - Y^2,\ X^2 + Y^2.$$

Solution. An elementary solution can be obtained by "completing squares". Let $a \neq 0$ or $c \neq 0$. Switching if necessary the roles of x and y, we may assume without loss of generality that $a \neq 0$. We rewrite: $ax^2 + 2bxy + cy^2 = a(x + by/a)^2 + (ac - b^2)y^2/a$. Assuming that $ac \neq b^2$ and taking $X = \sqrt{|a|}(x + \frac{by}{a})$ and $Y = \sqrt{\frac{|ac-b^2|}{|a|}}y$ for the new variables, we transform the quadratic form to one of the forms $\pm X^2 \pm Y^2$. When $ac = b^2$, we have $ax^2 + 2bxy + cy^2 = \pm X^2$ with the same X as above and $Y = y$. If both $a = 0$ and $c = 0$, but $b \neq 0$, we put $x = (x' + y')$, $y = (x' - y')$ and reduce the problem to the previous case. The remaining case $a = b = c = 0$ is among the 6 normal forms.

Finally, to see that the 6 normal forms are non-equivalent, we note that they have the following characteristic geometric properties which are preserved by linear changes of coordinates: $(X^2 + Y^2)$ is positive outside the origin, $-X^2 - Y^2$ is negative outside the origin, $\pm(X^2 - Y^2)$ (which are equivalent to each other by switching the roles of X and Y) are positive in some sector and negative in another one, X^2 is positive outside a line, $-Y^2$ negative outside a line, and 0 is zero everywhere.

Homework 8. Due by Tuesday, March 29.

1. Classify plane curves given by quadratic equations

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

(a) up to rigid motions of the plane;
(b) up to affine (i.e. linear inhomogeneous) transformations on the plane.

Solution. By the orthogonal diagonalization theorem, the quadratic function $ax^2 + 2bxy + cz^2 + dx + ey + f$ can be brought to the form $AX^2 + CY^2 + DX + EY + F$ by a suitable rotation of the coordinate system. When $A \neq 0$ (resp. $C \neq 0$) the linear term DX (resp. EY) can be eliminated by translations of the origin ("completing squares"). Using also the operations of renaming X and Y, changing their signs, and dividing the function by a non-zero constant, one can bring the equation of the curve to one of the following *normal forms*:
$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ (ellipse with semiaxes $\alpha \geq \beta > 0$,
$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ (hyperbola), $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 0$ (intersecting lines)
$y = \frac{x^2}{2\alpha^2}$ (parabola), $\frac{x^2}{2\alpha^2} = 1$ (two parallel lines), $\frac{x^2}{2\alpha^2} = 0$ (a double line)
$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 0$ (a point), $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = -1$ or $\frac{x^2}{2\alpha^2} = -1$ (the empty set).

If also rescaling of the coordinates is allowed, the equations of quadratic curves fall into fewer equivalence classes:

$X^2 + Y^2 = 1$ (circle), $= 0$ (point), $= -1$ (empty), $X^2 - Y^2 = 1$ (hyperbola), $= 0$ (intersecting lines), $X^2 = Y$ (parabola), $X^2 = 1$ (parallel lines), $= 0$ (a double line), $= -1$ (empty).

2. Show that the directions of meridians and parallels at every point on a surface of revolution are principal, and compute the Gaussian curvature of the surface obtained by rotating the graph of the function $x = f(z)$ about the z-axis.

Solution. The reflection in the plane passing through the axis of revolution and a meridian is an isometry of the surface of revolution and preserves the meridian pointwise. Thus at each point of the meridian this reflection preserves the 1st and the 2nd fundamental forms. Therefore the direction of the meridian is a principal axis of this pair of quadratic forms. The direction of the parallel is also principal as a direction perpendicular to a principal one. The curvatures $k_1(z)$ and $k_2(z)$ of the meridian and the parallel are $k_1 = f''(z)(1 + f'(z)^2)^{-3/2}$ and $k_2 = 1/f(z)$ and thus the Gaussian curvature is

$$K = k_1 k_2 = \frac{f''(z)}{f(z)(1 + f'(z)^2)^{3/2}}.$$

3. Let C be a regular curve on the surface of the unit ball, and let S be the cone over C with the vertex at the center of the ball. Find principal directions, principal curvatures and the Gaussian curvature of S at regular points.

Solution. The cone is *developable*, i.e. can be isometrically developed to the plane (or, equivalently, "made of a piece of paper"). Thus its Gaussian curvature $K = 0$ everywhere. By the Gauss Theorem Egreguim, at least one of the principal curvatures k_1, k_2 at each point equals 0. Therefore at a typical point of the cone (where, say, $k_1 \neq 0, k_2 = 0$) curves of all directions except the 2nd principal one have non-zero normal curvature. It is clear now that the 2nd principal direction is the direction of the *generator* of the cone (= the ray from the center of the ball) passing through this point (because straight lines in the space have zero normal curvature on any surface containing them). Thus the rays form one family of *curves of curvature*, i.e. curves having principal direction at each point, while the intersections of the cone with concentric spheres form the other such family (because the latter curves are everywhere perpendicular to the former ones).

4. Show that a regular surface near a non-umbilical point (i.e. a point where the two principal curvatures are distinct) posesses a local coordinate system such that both the 1st and the 2nd fundamental forms are diagonal:

$$I = A(u,v)(du)^2 + C(u,v)(dv)^2, \quad II = a(u,v)(du)^2 + b(u,v)(dv)^2.$$

Solution. In a sufficiently small neighborhood of a non-umbilical point, there are exactly 2 perpendicular principal directions in each tangent plane to the surface. The directions form two smooth *fields of directions* on the surface. By the Uniqueness and Existence Theorem for solutions of Ordinary Differential Equations, the surface is locally filled-in with a 1-parametric family of non-intersecting curves tangent to a given field of directions. Applying this to the two fields of principal directions we conclude that the surface is lically filled-in with two perpendicular 1-parametric families of curves of curvature (see Solution to Problem 3). Let u and v be parameters parameterizing curves of curvature of each of the two families. The fact that the directions $du = 0$ and $dv = 0$ are perpendicular means that the 1st fundamental form contains no $dudv$-term, and the fact that the directions are principal implies that the 2nd fundamental form contains no $dudv$-term.

Homework 9. Due by Tuesday, April 5.

1. Let V, E, F denote the numbers of vertices, edges, faces of a combinatorial surface (i.e. a "polyhedron" whose edges and faces are allowed to be curved), and let $\chi = V - E + F$ denote its *Euler characteristics*.

(a) Verify additivity of the Euler chracteristics in the following form: let X and Y be two combinatorial surfaces whose intersection Z is a combinatorial sub-surface in each of them. Then $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(Z)$.

(b) Show that the Euler characteristics of closed regular surfaces (spheres, projective planes and Klein botles with g handles) are respectively

$$\chi(S_g^2) = 2 - 2g, \quad \chi(P_g^2) = 1 - 2g, \quad \chi(K_g^2) = -2g.$$

Solution. (a) In fact the additivity obviously holds true for the numbers of vertices, edges and faces separately: by adding these in X and Y we are counting twice those which are in both.

(b) Thanks to topological invariance of the Euler characteristic (for closed oriented surfaces it follows from the Gauss-Bonnet theorem), it suffices to compute $E - V + F$ for any particular combinatorial structure. Moreover, one can find this way that the Euler characteristics of the disc, circle, cyclinder and Möbius strip are equal respectively to $1, 0, 0, 0$ and then use additivity. Namely, sphere is 2 discs glued along a common circle, thus $\chi(S^2) = 1 + 1 - 0 = 2$. Similarly (see the next problem), $\chi(P^2) = 1 + 0 - 0 = 1$ (disc plus the Möbius strip minus circle), and $\chi(K^2) = 0 + 0 - 0$ (2 Möbius strips glued along a circle). Adding a handle consists of detaching 2 discs and attaching a cylinder along 2 circles $(-2 + 0 - 0 - 0)$ and thus decreases the Euler characteristic by 2.

2. Show that gluing a disc and a Möbius strip along their boundaries results in the projective plain P^2. Identify the surface obtained by gluing two Möbius strips along their boundaries.

Solution. Cutting a Möbius strip along the middle circle yields a cylinder. The reverse procedure consists in identifying pairs of opposite points on *one* of the boundaries of the cylinder. Attaching a disc to the other boundary results in a (larger) whose boundary's opposite points to be indetified. This is one of our models of P^2 which is thus glued from a Möbius strip and a disc.

Just like any other bottle, Klein's one has a plane of symmetry. Cutting it in half along this plane yields two "half-bottles". Each one is a rectangular strip with two opposite sides identified in a perverse way, i.e. as in the Möbius strip.

3. Let Σ be a connected regular surface containing a Möbius strip. Show that detaching 2 discs from Σ and replacing them with 2 Möbius strips is equivalent to attaching a handle to Σ (i.e. that the resulting two surfaces are homeomorphic).

Solution. We may assume that the 3 Möbious strips M_1, M_2, M_3 are attached near 3 points p_1, p_2, p_3 which belong to the same disk $D \subset \Sigma$. Let us instead attach to D a handle H near p_1 and p_2. The resulting surface remains homeomorphic to itself when the point p_2 is moving continuously on the surface. We let it move along the middle circle of M_3 and then come back to the original position p_2. Since M_3 is one-sided, the resulting handle H is now attached to D near p_1 and p_2 from opposite sides. Thus D^2 (which can be thought of as the sphere minus a disc) together with this disorienting handle is the same as K^2 minus the disc. Let us now think of K^2 as M_1 and M_2 glued along an intermediate cylinder. Detaching the disc from the cylinder results in a disc with two holes glued in by M_1 and M_2. Reminding ourselves about M_3, we conclude that the disc D with M_3 and H is homeomorphic to the disc D with M_1, M_2 and M_3.

4. Show that any closed regular surface in $\mathbb{R}^3$ has elliptic points. Can it have no hyperbolic, parabolic points? How does the answer depend on the genus?

Solution. Since the surface is compact, it contains a point p furthest from the origin. The sphere centered at the origin and passing through p encloses the surface and touches it at p. Respectively, the paraboloid osculating the sphere at p encloses the paraboloid osculating the surface at p. Thus both principal curvatures of the surface at p are minorated by $1/r > 0$, where r is the radius of the sphere. Therefore p is an elliptic point on the surface, i.e. $K(p) > 0$.

On the other hand, the Gauss-Bonnet theorem guarantees that the integral $\int\int K dA = 4\pi(1-g) \leq 0$ when the genus g of the surface is positive. Since $K > 0$ in a neighborhood of p, K must be negative elsewhere and pass through 0 somewhere in between. Thus a closed surface of genus $g > 0$ must have elliptic, hyperbolic and parabolic points.

The example of the standard sphere shows that a closed genus 0 surface may have all points elliptic.

Homework 10. Due by Thursday, April 14.

1. *Angular defects of all vertices of a convex polyhedron* (with linear edges and faces this time) *in* $\mathbb{R}^3$ *add up to* 4π*.* Prove this statement and its generalization to polyhedra homeomorphic to other (than S^2) closed surfaces following the argument in the proof of the Gauss-Bonet theorem.

Solution. Let the faces of the polyhedron be n_i-gons P_i, $i = 1, ..., F$. The angle sum of P_i is $\pi(n_i - 2)$. Summing all angles of all the faces we therefore get $\pi \sum n_i - 2\pi F$. The face P_i has n_i sides, and each out of E edges of the polyhedron occurs as a side in exactly 2 faces. Thus $\pi \sum n_i = 2\pi E$. On the other hand, all the angles groupped by the V vertices of the polyhedron with angular defects $\delta_1, ..., \delta_V$, add up to $\sum(2\pi - \delta_j) = 2\pi V - \sum \delta_j$. Thus the total sum of angular defects

$$\sum_{j=1}^{V} \delta_j = 2\pi(V - E + F) = 2\pi\chi.$$

2. (a) Express the total Gaussian curvature of a Riemannian metric on the disc D in terms of the geodesic curvature of the boundary ∂D.
(b) Generalize the Gauss-Bonnet theorem to compact surfaces with boundaries.

Solution. Using the result of Problem 3 from Homework 6 and following the proof of the Gauss–Bonnet theorem for closed surfaces based on partitioning the surface into F faces by E curvilinear edges and with V vertices, we obtain

$$\int\int_{\Sigma} KdArea + \int_{\partial\Sigma} k_g dLength = 2\pi(V - E + F) = 2\pi\chi(\Sigma).$$

3. Show that the total Gaussian curvature of the surface $x^2 + y^2 - z^2 = r^2$ does not depend on r and compute it.

Solution. We apply the result of the previous problem to the surface $Sigma_M$ with boundary which is the part of the hyperboloid $x^2 + y^2 - z^2 = r^2$ satisfying $|z| \le M$, and pass to the limit $M \to \infty$. Since the $\chi(\Sigma_M) = 0$, we have $\int_{\Sigma M} KdArea = -\oint_{\partial\Sigma_M} k_g ds$. When $z \to \pm\infty$, the hyperboloid asymptotically approaches the cone $x^2 + y^2 = z^2$. Developing the cone to the plane, we find that the geodesic curvature of each circle $z = \pm M$ on it is equal to $1/\sqrt{2}M$. Thus when $M \to \infty$ the total geodesic curvature of $\partial\Sigma_M$ tends to $2 \times 2\pi M/\sqrt{2}M = 2\sqrt{2}\pi$. Thus the total Gaussian curvature of the hyperboloid is equal to $-2\sqrt{2}\pi$, and in particular does not depend on r.

4. Let $P(z)$ and $Q(z)$ be relatively prime polynomials of degree p and q with complex coefficients. Consider the rational function $P(z)/Q(z)$ of a complex variable as a map from $S^2 = \mathbb{C} \cup \infty$ to $S^2 = \mathbb{C} \cup \infty$ and compute the degree of this map.

Hint: First consider the operation $\mathbb{C} \to \mathbb{C} : z \mapsto (a+bi)z$ of multiplication by a given complex number as a linear map from the plane $\mathbb{C} = \mathbb{R}^2$ to itself and compute its determinant.

Solution. Expanding the rational function $F = P/Q$ near a point z we find

$$F(z + \Delta z) = F(z) + F'(z)\Delta z + o(\delta z).$$

Thus the linearization of the map $z \mapsto F(z)$ at the point z is the linear map $\mathbb{C} \to \mathbb{C}$ given by $\Delta z \mapsto F'(z)\Delta z$, i.e. multiplication by the complex number $a + bi = F'(z)$. Following the hint, we find in real coordinates $\Delta z = x + yi$ the linear map is $(x, y) \mapsto (ax - by, bx + ay)$ with the determinant $a^2 + b^2$. Thus the Jacobian determinant of the map given by F is non-negative (and vanishes only when $F'(z) = 0$). This implies that each preimage $z \in F^{-1}(c)$ of a regular value c contributes $+1$ to the degree of the map. To find the number of preimages we need to solve $P(z)/Q(z) = c$, or equivalently $P(z) - cQ(z) = 0$. This is a polynomial equation of degree $m = \max(p, q)$. By the Fundamental Theorem of Algebra (and Sard's Lemma) it has m simple distinct solutions for almost all c. Thus the degree of the map is $m = \max(p, q)$.

Homework 11. Due by Thursday, April 21.

1. Describe the spherical image of the surface $x^2 + y^2 - z^2 = r^2$ under the Gauss map and find its signed area. Compare the result with the total Gaussian curvature found in problem 1 of the previous homework.

Solution. The surface is a one-sheeted hyperboloid of revolution asymptitically approaching the cone $x^2 + y^2 = z^2$ whose generators make the angle $\pi/4$ with the axis z of revolution. The normal unit vectors to the cone form two circles $z = \pm 1/\sqrt{2}$ on the unit sphere $x^2+y^2+z^2 = 1$. By inspection, the region $|z| < 1/\sqrt{2}$ on the unit sphere between these two circles is the spherical image of the hyperboloid, while the Jacobian of the Gauss map is negative. Thus the signed area of the spherical image is equal to $-4\pi/\sqrt{2} = -2\sqrt{2}\pi$. This agrees with result of Problem 1 from Homework 10.

2. *Angular defects of all vertices of a convex polyhedron* (with linear edges and faces this time) *in* $\mathbb{R}^3$ *add up to* 4π. Prove this statement by mimicking the proof of Gauss-Bonnet theorem based on the Gauss map.

Solution. Consider the unit exterior normal vectors f_i to the faces of the polyhedron as points on the unit sphere. When two faces (say i and j) share an edge, connect f_i and f_j with an equatorial arc on the unit sphere (the arc is perpendicular to the edge and imitates the range of the Gauss map for the surface obtained from the two faces by smoothening over the edge). The construction results in a combinatorial structure on the unit sphere *dual* to the original polyhedron P. The vertices are f_i; they correspond to faces of P. The equatorial arcs connecting the vertices correspond to the edges of P. The arcs partition the sphere into regions which correspond to the vertices of P. (The regions mimick spherical images of neighborhoods of the vertices after smoothening of P). We claim that *the spherical area of each region is equal to the angular defect of the corresponding vertex of* P. Since the regions add up to the whole sphere, this would imply that the angular defects add up to 4π.

To justify the claim, consider a polyhedral cone with faces (in cyclic order) $F_1, ..., F_n$ and unit exterior normal vectors $f_1, ..., f_n$. The plane spanned by f_{i-1} and f_i is perpendicular to the common ray of F_{i-1} and F_i, and similarly the plane of f_i and f_{i+1} is perpendicular to the common ray of F_i and F_{i+1}. This implies that the angles between these two planes and between these two rays add up to π. Considering now the spherical n-gon formed by the vertices f_i, we find $\alpha_i+\beta_i = \pi$, where

α_i is the angle at the vertex f_i of the n-gon, and β_i is the angle in the face F_i of the cone. By the local Gauss-Bonnet theorem for spherical polygons we find that the area of the spherical n-gon $f_1 ... f_n$ is

$$Area = \sum \alpha_i - \pi(n-2) = 2\pi - \sum \beta_i = \delta,$$

i.e. exactly the angular defect of the polyhedral cone.

3. (a) Show that the torus and the Klein bottle can be equipped with a Riemannian metric of zero curvature.

(b) Are all tori of zero Gaussian curvature isometric to each other?

Solution. (a) Gluing tori and Klein bottles from a square can be understood as factorizaion of the Euclidean plane $\mathbb{R}^2$ by the action of a group of rigid motions generated (for tori) by two non-colinear translations, and (for Klein bottles) by a translation and another translation composed with reflection, e.g. $(x, y) \mapsto (x, y+1)$ and $(x, y) \mapsto (x+1, -y)$. The resulting quotient surface inherits from $\mathbb{R}^2$ a Riemannian metric of zero Gaussian curvature.

(b) Multiplying the metric in $\mathbb{R}^2$ by a constant we equip the quotient torus with a zero curvature metric of a different total area. [A less obvious way to construct non-isometric zero curvature tori is to glue them from Euclidean parallelograms (rather than squares) of different shapes.]

4. In the Minkovsky 3-space, consider three planes passing through the origin and intersecting pairwise along three lines situated on the light cone. The there planes cut out a triangle on the "upper" sheet of the Minkovsky sphere of radius R. Find the Riemannian area of this triangle.

Solution. By the local Gauss-Bonnet theorem, the total curvature KA of a geodesic triangle with the angles α, β, γ and area A on a surface of constant Gaussian curvature K is equal to $\alpha + \beta + \gamma - \pi$. On the Minkovsky sphere of radius R, we have $K = -1/R^2$, and thus $A = R^2(\pi - \alpha - \beta - \gamma)$. The "tiangle" described in the problem has its vertices at infinity and is the limiting case of a geodesic triangle with $\alpha, \beta, \gamma \to 0$. Thus $A = \pi R^2$.

Homework 12. Due by Thursday, April 28.

1. Show that all plane elements tangent to a given space curve C form an integral surface $L_C \subset Q^5$ in the quadric of all plane elements.

Solution. Let $(v(t), \omega(t))$ be a family of plane elements tangent to C. Then $\omega(t)$ is a plane containing the tengent line to C at the point $v(t) \in C$, and $\dot{v}(t)$ is tangent to C at $v(t)$. Thus $\dot{v}(t) \in \omega(t)$, as required in the definiton of integral curves and integral surfaces.

2. A surface is called *ruled* if together with each point it contains a straight line passing through this point. Prove that the surface $S^*_C \subset V^*$, obtained from the curve $C \subset V$ by projecting to V^* the corresponding integral surface $L_C \subset Q^5 \subset V \times V^*$, is ruled.

Solution. The projection S^*_C consists of all planes tangent to C. Let ω be such a plane, tangent to C at v. Then it containes the line l tangent to C at v. All planes containing $l \subset V$ form a line $l^* \subset V^*$. This line is contained in S^*_C and passes through $\omega \in S^*_C$.

3. Let the space curve C in Problem 1 be the circle $x^2 + y^2 = r^2, z = 1$. Let $L_C \subset Q^5$ be the corresponding integral surface $L_C \subset Q^5$. Compute the surface $S^*_C \subset V^*$ obtained by projecting of L_C from $Q^5 \subset V \times V^* \to V^*$. Show that S^*_C is a quadratic surface and indentify it.

Solution. A plane $\alpha x + \beta y + \gamma z = 1$ is tangent to the circle at the point $(x, y, z) = (r \cos t, r \sin t, 1)$ if $\alpha r \cos t + \beta r \sin t + \gamma = 1$, and $\beta r \cos t - \alpha r \sin t = 0$. To eliminate t, we square and add the equations, and obtain $r^2\alpha^2 + r^2\beta^2 = (\gamma - 1)^2$. This surface in the (α, β, γ)-space is a cone of revolution about the γ-axis with the vertex $(\alpha, \beta, \gamma) = (0, 0, 1)$.

4. Let $S \subset V = \mathbb{R}^3$ be a quadratic surface given by the equation $(Av, v) = 1$ where A is a symmetric 3×3-matrix, and $(\cdot, \cdot)$ is the dot-product in $\mathbb{R}^3$. Compute the dual surface $S^* \subset V^*$.

Solution. The differential of the quadratic function (Av, v) is $(Av, dv) + (Adv, v) = 2(Av, dv)$. Therefore the plane given by the equation $(\omega, x) = 1$ with $\omega = Av$ is tangent to S at v. Thus the equation $(Av, v) = 1$ for $v = A^{-1}\omega$ transforms into the equation $(\omega, A^{-1}\omega) = 1$ of the dual surface S^*.

Homework 13. Due by Thursday, May 5.

1. Show that a ruled surface cannot have positive Gaussian curvature, and give an example of a ruled surface of negative Gaussian curvature.

Solutions. A straight line on a regular surface has zero normal (as well as any other) curvature. Therefore the 2nd fundamental form of a ruled surface cannot be sign-definite. A hyperboloid of one sheet is an example of a ruled surface which has *negative* Gaussian curvature (e.g. because there are *two* straight lines through each point).

2. A regular curve C on a regular surface $S \subset \mathbb{R}^3$ is called *asymptotic* if the 2nd fundamental form of S vanishes on the tangent lines to C.

(a) Show that each point on a hyperbolic ($K < 0$) surface is contained in two asymptotic curves.

(b) Prove that the geodesic curvature of an asymptotic curve C on a surface coincides with the curvature of C considered as a space curve.

Solution. (a) The 2nd fundamental form of a hyperbolic surface vanishes on a crossing pair of tangent lines. This defines locally on the surface two fields of asymptotic directions. By the Existence and Uniqueness Theorem for solutions of Ordinary Differential Equations, each of the fields of directions can be locally integrated: the surface is filled-in with a 1-parametric family of non-intersecting curves everywhere tangent to the field of directions. The curves of each family are therefore asymptotic.

(b) The normal curvature k_n of a curve on a surface is given by the value of the 2nd fundamental form on the unit velocity vector of the curve. By definition, asymptotic curves have therefore $k_n = 0$. The relationship $k_n^2 + k_g^2 = k^2$ between normal, geodesic, and space curvature of the curve shows that asymptotic curves have $|k_g| = k$.

3. Show that if a regular curve C is asymptotic on a regular surface S, then C^* is contained in S^* and is asymptotic on S^* (at non-singular points). Vice versa, when the images in S and S^* of the same curve in $L_S = L_{S^*} \subset Q^5$ happen to be dual curves C and C^*, then these curves are asymptotic respectively on S and S^*.

Solution. By Problem 2, the acceleration vector of a naturally parameterized space curve C asymptotic on a regular surface S has no normal component. Therefore osculating planes of C are tangent to S. Thus the dual curve $C^* \subset V^*$ formed by the osculating planes of C is contained in the dual surface S^* formed by tangent planes to S. Vice versa, the points representing $C \subset S$ on the dual surface are tangent planes to S at C. If they form the dual curve C^*, this means that tangent planes to S at C are osculating planes of C. In other words, the acceleration vectors of C are tangent to S, i.e. the normal curvature of C vanishes, and therefore C is asymptotic on S. Using $C = C^{**}$ and reversing the roles of C and C^* we conclude that under the circumstances C^* is asymptotic on S^* as well.

4. Describe regular surfaces $S \subset V$ whose tangent planes form in V^* a plane curve.

Solution. Let $P^* \subset V^*$ be a plane passing through the origin. The condition $\langle v, \omega \rangle = 0$ for all $\omega \in P$ determines a line $L \subset V$ passing through the origin. Next, planes containing the line $l^* \subset P^* - 0$ passing through two points $\omega_1, \omega_2 \in l^*$ form the line $l \subset V - 0$ determined by two equations $\langle v, \omega_1 \rangle = \langle v, \omega_2 \rangle = 1$. The line l is parallel to L because for $v_1, v_2 \in l$ we have $\langle v_1 - v_2, \omega_i \rangle = 0$ for $i = 1, 2$. When tangent planes to $S \subset V$ form a curve $C^* \subset P^*$, the surface $S_{C^*} \subset V$ containing S is formed by those planes in V^* which contain a tangent line to C^*. Thus S_{C^*} is ruled by lines l parallel to L and is a cylindrical surface in V.

Suppose now that tangent planes to S form a curve C^* contained in a plane P' not passing through the origin (and is given therefore by the equation $\langle v', \cdot \rangle = 1$). The surface $S_{C^*} \supset S$ is still ruled by straight lines l formed by planes in V^* containing a tangent line l^* to C^*. Since P' is one of such planes, the lines l pass through v'. Thus S_{C^*} is a conical surface with the vertex v'.