Math 140. Homework.

Homework 1. Due by Tuesday, 01.25.05

1. Let D_d be the family of domains in the Euclidean plane bounded by the smooth curves ∂D_d equidistant to a bounded convex domain D_0. How does the perimeter $Length(\partial D_d)$ depend on the distance d between ∂D_d and D_0?

2. Verify the invariance of the arc length $\int_a^b \sqrt{\dot{x}^2(t)+\dot{y}^2(t)}\, dt$ under reparameterizations $t = t(\tau)$.

3. (a) Prove the formula $k = (\ddot{x}\dot{y} - \ddot{y}\dot{x})/(\dot{x}^2+\dot{y}^2)^{3/2}$ for the curvature of a regular parameterized plane curve $t \mapsto (x(t), y(t))$.

(b) Compute the curvature of the graph of a smooth function $y = f(x)$.

(c) Take $f = x^a/a$ and find the limit of curvature at $x = 0$ for $a = 5/2, 2, 3/2, 1, 1/2$.

4. Draw the typographic symbol ∞ ("infinity" or "figure eight") increased 100 times and then draw an equidistant curve as follows: orient all normal lines to the large figure eight in a continuous fashion, and connect all points removed 1 *cm* from the large figure eight in the *positive* normal direction. Which curve is longer — the large figure eight or the curve equidistant to it?

Homework 2. Due by Tuesday, 02.01.05

1. Show that maps $\mathbb{R}^2 \to \mathbb{R}^2 : \mathbf{x} \mapsto \mathbf{y}$ which preserve all Euclidean distances are given by linear inhomogeneous functions, namely by compositions of translations with rotations or reflections.

[Hint: Show that two isometries of the plane coincide if they act the same way on the vertices of your favorite triangle.]

2. Compute the curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $(x_0, y_0) = (0, b)$.

3. Let $t \mapsto (x(t), y(t))$ be a *closed* regular plane curve. Let $t \mapsto (\dot{x}(t), \dot{y}(t))$ be the closed regular plane curve formed by the velocity vectors. Prove that the integral

$$\frac{1}{2\pi} \oint \frac{\dot{x} d\dot{y} - \dot{y} d\dot{x}}{\dot{x}^2 + \dot{y}^2}$$

is an integer.

[Hint: Use Green's theorem from Multivariable Calculus.]

Point out geometric interpretations of this integer in terms of the velocity curve and of the original curve.

4. Compute the cutvature and torsion of the parameterized space curves (t, t^2, t^3), (t, t^2, t^4), (t, t^3, t^4) at $t = 0$.

Homework 3. Due by Thursday, 02.10.05

1. Prove that a space curve with the identically zero torsion is contained in a plane.

2. The inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is related to the length $\| \cdot \|$ by means of the *polarization identity*:

$$\langle x, y \rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2).$$

Prove this identity, and deduce from it that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is any length-preserving linear transformation, then T preserves the inner product, i.e.

$$\langle T(x), T(y) \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^n$.

3. Let $A(t)$ be an anti-symmetric $n \times n$-matrix depending continuously on t, and U_0 be an orthogonal $n \times n$-matrix (i.e. $A^* = -A$, and $U_0^* = U_0^{-1}$, where * means transposition).

Consider the system $\dot{M} = A(t)M$ of n^2 linear ordinary differential equations in the space $\mathbb{R}^{n^2}$ of $n \times n$-matrices M.

Prove that the solution $t \mapsto M(t)$ to this system satisfying the initial condition $M(t_0) = U_0$ consists of orthogonal matrices $M(t)$.

4. Does there exist a *closed* space curve with constant nonzero curvature and (somewhere) nonzero torsion?

Homework 4. Due by Thursday, 02.17.05

1. For each of the 5 Platonic solids (tetrahedron, cube, octahedron, icosahedron and dodecahedron), compute the *angular defect* at each vertex, i.e. the difference between 2π and the sum of face's angles adjecent to this vertex. What do the angular defects of all vertices add up to?

2. Given two skew-lines in $\mathbb{R}^3$ (i.e. two straight lines which are not parallel and have no common points), rotate one of them about the other, find the equation of the resulting surface of revolution and show that the surface is a hyperboloid of one sheet.

Monic degree-3 polynomials $P(x) = x^3 + ax^2 + bx + c$ form a 3-dimensional space with coordinates (a, b, c). In this space, consider the *discriminant* Δ — the surface formed by those polynomilas which have a multiple root. Such polynomials have the form $P(x) = (x-u)^2(x-v)$ which provides a parameterization of Δ by (u, v).

3. (a) Sketch the section of the discriminant by the plane $a = 0$.

(b) Show that the transformation $P(x) \mapsto P(x+t)$ defines a (non-linear) flow in the space of polynomials which preserves Δ and transforms the plane $a = 0$ to $a = 3t$. Use this to sketch Δ.

4. (a) Show that singular points of Δ form the curve C consisting of polynomials $(x-a)^3$ with a triple root, and show that Δ is the *osculating surface* of C (i.e. is swept by tangent lines to C).

(b) Sketch the osculating surface of the curve (t, t^2, t^3) together with its osculating plane at $t = 0$. (*Hint:* the curve can be identified with C by stretching the coordinates.)

Homework 5. Due by Thursday, February 24.

1. Show that the Riemannian area

$$\int\int_D \sqrt{AC - B^2}\, dU dV$$

of a region D on the plane equipped with a Riemannian metric

$$A(U,V)(dU)^2 + 2B(U,V)(dU)(dV) + C(U,V)(dV)^2$$

is invariant with respect to changes of variables $U = U(u,v), V = V(u,v)$.

2. (a) Compute the Riemannian metric, induced by the standard embedding of the sphere of radius r into the Euclidean 3-space, in terms of spherical coordinates.

(b) Using (a) compute the Riemannian area of the spherical triangle bounded by the equator and by two meridians making the angle ϕ to each other.

3. In cartography, a popular way to obtain a plane image of the Earth's surface is based on the projecting the sphere $x^2 + y^2 + z^2 = r^2$ onto the cylinder $x^2 + y^2 = r^2, |z| \leq r$ by rays in the planes $z = const$ radiating away from the z-axis:

$$(x,y,z) \mapsto (xr/\sqrt{r^2 - z^2}, yr/\sqrt{r^2 - z^2}, z).$$

Show that this projection of the sphere to the cylinder preserves areas. Is this projection an isometry, i.e. preserves lengths of all curves?

4. Compute the geodesic curvature of the circle $z = r/2$ on the surface of the sphere $x^2 + y^2 + z^2 = r^2$ and compare it with the curvature of the same circle considered as a curve in the space.

Homework 6. Due by Tuesday, March 8.

1. Perform the parallel transport around the loop on the sphere $x^2 + y^2 + z^2 = r^2$ cut out by the plane $z = r/2$. Find the area enclosed by this curve on the sphere.

2. Prove that meridians on a surface of revolution are geodesics. Are all paralleles geodesics too?

3. Express the rotation angle under parallel transport around a curvilinear triangle on a Riemannian surface in terms of angles at the vertices of the triangle and the total geodesic curvature of its sides.

4. Prove the n-dimensional version of the Key Lemma: *Any Riemannian metric in* $\mathbb{R}^n$ *near any point is Euclidean modulo terms of order* ≥ 2 *in a suitable local coordinate system, which is unique up to linear orthogonal transformations and modulo terms of order* ≥ 3. Deduce that Riemannian metrics do not have local invariants depending only on the derivatives of the metric of order ≤ 1.

Homework 7. Due by Tuesday, March 15.

1. Show that the Gaussian curvature $K(p)$ at a point p on a Riemann surface depends only on the derivatives of order ≤ 2 of the Riemannian metric at this point.

2. Let us call a coordinate system (u, v) on a Riemann surface *Gaussian* with respect to the point $(0, 0)$ if the Riemannian metric has the form

$$(du)^2 + (dv)^2 - \frac{K}{3}(udv - vdu)^2 + \text{ (terms of order } \geq 3).$$

Show that the coefficient K equals the Gaussian curvature of this Riemannian metric at the origin.

3. Show that in a Gaussian coordinate system geodesics passing through the origin coinside with straight lines up to order 3, i.e. not only have $r''(0) = 0$ but also $r'''(0) = 0$.

Hint: check that any straight line through the origin is a symmetry line of the metric $(du)^2 + (dv)^2 - K(udv - vdu)^2/3$ (without higher order terms).

4. Show that any quadratic form $ax^2 + 2bxy + cy^2$ on the plane can be transformed by a linear change of coordinates to one and only one of the following six forms

$$0,\ X^2,\ -Y^2,\ -X^2 - Y^2,\ X^2 - Y^2,\ X^2 + Y^2.$$

Homework 8. Due by Tuesday, March 29.

1. Classify plane curves given by quadratic equations

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

(a) up to rigid motions of the plane;

(b) up to affine (i.e. linear inhomogeneous) transformations on the plane.

2. Show that the directions of meridians and parallels at every point on a surface of revolution are principal, and compute the Gaussian curvature of the surface obtained by rotating the graph of the function $x = f(z)$ about the z-axis.

3. Let C be a regular curve on the surface of the unit ball, and let S be the cone over C with the vertex at the center of the ball. Find principal directions, principal curvatures and the Gaussian curvature of S at regular points.

4. Show that a regular surface near a non-umbilical point (i.e. a point where the two principal curvatures are distinct) posesses a local coordinate system such that both the 1st and the 2nd fundamental forms are diagonal:

$$I = A(u,v)(du)^2 + C(u,v)(dv)^2, \;\; II = a(u,v)(du)^2 + b(u,v)(dv)^2.$$

Homework 9. Due by Tuesday, April 5.

1. Let V, E, F denote the numbers of vertices, edges, faces of a combinatorial surface (i.e. a "polyhedron" whose edges and faces are allowed to be curved), and let $\chi = V - E + F$ denote its *Euler characteristics.*

(a) Verify additivity of the Euler chracteristics in the following form: let X and Y be two combinatorial surfaces whose intersection Z is a combinatorial sub-surface in each of them. Then $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(Z)$.

(b) Show that the Euler characteristics of closed regular surfaces (spheres, projective planes and Klein botles with g handles) are respectively

$$\chi(S^2_g) = 2 - 2g, \quad \chi(P^2_g) = 1 - 2g, \quad \chi(K^2_g) = -2g.$$

2. Show that gluing a disc and a Möbius strip along their boundaries results in the projective plain P^2. Identify the surface obtained by gluing two Möbius strips along their boundaries.

3. Let Σ be a connected regular surface containing a Möbius strip. Show that detaching 2 discs from Σ and replacing them with 2 Möbius strips is equivalent to attaching a handle to Σ (i.e. that the resulting two surfaces are homeomorphic).

4. Show that any closed regular surface in $\mathbb{R}^3$ has elliptic points. Can it have no hyperbolic, parabolic points? How does the answer depend on the genus?

Homework 10. Due by Thursday, April 14.

1. *Angular defects of all vertices of a convex polyhedron* (with linear edges and faces this time) *in* $\mathbb{R}^3$ *add up to* 4π. Prove this statement and its generalization to polyhedra homeomorphic to other (than S^2) closed surfaces following the argument in the proof of the Gauss-Bonet theorem.

2. (a) Express the total Gaussian curvature of a Riemannian metric on the disc D in terms of the geodesic curvature of the boundary ∂D.

(b) Generalize the Gauss-Bonnet theorem to compact surfaces with boundaries.

3. Show that the total Gaussian curvature of the surface $x^2 + y^2 - z^2 = r^2$ does not depend on r and compute it.

4. Let $P(z)$ and $Q(z)$ be relatively prime polynomials of degree p and q with complex coefficients. Consider the rational function $P(z)/Q(z)$ of a complex variable as a map from $S^2 = \mathbb{C} \cup \infty$ to $S^2 = \mathbb{C} \cup \infty$ and compute the degree of this map.

Hint: First consider the operation $\mathbb{C} \to \mathbb{C} : z \mapsto (a + bi)z$ of multiplication by a given complex number as a linear map from the plane $\mathbb{C} = \mathbb{R}^2$ to itself and compute its determinant.

Homework 11. Due by Thursday, April 21.

1. Describe the spherical image of the surface $x^2+y^2-z^2=r^2$ under the Gauss map and find its signed area. Compare the result with the total Gaussian curvature found in problem 1 of the previous homework.

2. *Angular defects of all vertices of a convex polyhedron* (with linear edges and faces this time) *in* $\mathbb{R}^3$ *add up to* 4π. Prove this statement by mimicking the proof of Gauss-Bonnet theorem based on the Gauss map.

3. (a) Show that the torus and the Klein bottle can be equipped with a Riemannian metric of zero curvature.

(b) Are all tori of zero Gaussian curvature isometric to each other?

4. In the Minkovsky 3-space, consider three planes passing through the origin and intersecting pairwise along three lines situated on the light cone. The there planes cut out a triangle on the "upper" sheet of the Minkovsky sphere of radius R. Find the Riemannian area of this triangle.

Homework 12. Due by Thursday, April 28.

1. Show that all plane elements tangent to a given space curve C form an integral surface $L_C \subset Q^5$ in the quadric of all plane elements.

2. A surface is called *ruled* if together with each point it contains a straight line passing through this point. Prove that the surface $S^*_C \subset V^*$, obtained from the curve $C \subset V$ by projecting to V^* the corresponding integral surface $L_C \subset Q^5 \subset V \times V^*$, is ruled.

3. Let the space curve C in Problem 1 be the circle $x^2 + y^2 = r^2, z = 1$. Let $L_C \subset Q^5$ be the corresponding integral surface $L_C \subset Q^5$. Compute the surface $S^*_C \subset V^*$ obtained by projecting of L_C from $Q^5 \subset V \times V^* \to V^*$. Show that S^*_C is a quadratic surface and indentify it.

4. Let $S \subset V = \mathbb{R}^3$ be a quadratic surface given by the equation $(Av, v) = 1$ where A is a symmetric 3×3-matrix, and $(\cdot, \cdot)$ is the dot-product in $\mathbb{R}^3$. Compute the dual surface $S^* \subset V^*$.

Homework 13. Due by Thursday, May 5.

1. Show that a ruled surface cannot have positive Gaussian curvature, and give an example of a ruled surface of negative Gaussian curvature.

2. A regular curve C on a regular surface $S \subset \mathbb{R}^3$ is called *asymptotic* if the 2nd fundamental form of S vanishes on the tangent lines to C.

(a) Show that each point on a hyperbolic ($K < 0$) surface is contained in two asymptotic curves.

(b) Prove that the geodesic curvature of an asymptotic curve C on a surface coincides with the curvature of C considered as a space curve.

3. Show that if a regular curve C is asymptotic on a regular surface S, then C^* is contained in S^* and is asymptotic on S^* (at non-singular points). Vice versa, when the images in S and S^* of the same curve in $L_S = L_{S^*} \subset Q^5$ happen to be dual curves C and C^*, then these curves are asymptotic respectively on S and S^*.

4. Describe regular surfaces $S \subset V$ whose tangent planes form in V^* a plane curve.