Answers for HW1:

1. Let $\psi : V^n \to X \subset \mathbb{R}^N$ be the local parametrization of X near $x \in X$, and let $f : X \to Y \subset \mathbb{R}^M$ be smooth in the sense (ii). By the definition of local parametrization, it is a diffeomorphism onto a neighborhood of x in X, and hence (by the definition of a diffeomorphism) the inverse map extends smoothly to a neighborhood U^N of x in the ambient space $\mathbb{R}^N$: $\phi : U^N \to V^n$. Thus $f \circ \psi \circ \phi$ is a local extension of f near x to a smooth map: $U^N \to \mathbb{R}^M$. Thus f is smooth in the sense (i). The converse should be obvious.

2. Let $z_i = a_i + b_i\sqrt{-1}$, $i = 1, 2, 3$. Then $\sum z_i^2 = 0$ and $\sum |z_i|^2 = 1$ are equivalent to $\sum a_i^2 = \sum b_i^2 = 1/2$, and $\sum a_i b_i = 0$, while $\sum |z_i|^2 = 1$. Thus, Y is the space of pairs a, b of orthogonal 3-dimensional vectors of fixed length, $1/\sqrt{2}$. Normalizing a, b to the unit length and adding the third vector $c = a \times b$ (the cross-product), we get a 3×3-matrix U (with the three vectors being the columns of U) such that $U^t U = I$ and $\det U = +1$. Obviously this mapping $f : Y \to X$ is invertible, and both f and f^{-1} are smooth.

3. The torus in $\mathbb{R}^3$ is obtained by rotating around the z-axis the circle $z^2 + (x - a)^2 = b^2$ in the plane $y = 0$. Thus a point on this serface of revoluion is determined by two angles: one on the rotated circle, the other being the angle of rotation. This identifies the torus with $S^1 \times S^1$.

4. The key fact is that $\lim_{y\to\infty} y^n/e^y = 0$ for any n. This can be derived by the application of l'Hospital's rule. Taking $y = 1/x^2$ we conclude by induction that the function $f := e^{-1/x^2}$ for $x > 0$ and extended by $f = 0$ for $x \le 0$ has all derivatives $f^{(n)}(0)$ well-defined (and equal 0 of course). Part (b) is straightforward, and (c) is solved by $1 - h(|x|)$.

5. In problem 8, $x^2 + y^2 - z^2 = a$ are: the cone when $a = 0$, hypoboloid of one sheet when $a > 0$, and hyperboloit of two sheets when $a < 0$. Calling the surfaces "paraboloids" is a mistake (repeated in problem 8 of section 2). Paraboloids are graphs of quadratic functions, e.g. $z = x^2 \pm y^2$.