Math H104. Take Home Final

Rules: Solve the following problems and submit your written solutions at 3:30 p.m. on Tuesday, December 9, 2008. You may bring your solutions to the class (or submit them in any other way, e.g. email to gvental@math.berkeley.edu).

While solving the problems, you may not consult other people or any sources (including Internet), with the exception of the textbook for this course.

Breaking the above rule constitutes a serious offense against the honor code thereby imposed on you, and may result in severe penalties. By submitting your final exam you acknowledge that you followed the rules.

1. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous compactly supported function (i.e. $f(x) = 0$ for all sufficiently large $x$). Define the $L$-transform$^1$ of $f$ as

$$
\hat{f}(z) = \int_0^\infty e^{-x} f(xz) dx.
$$

(a) Prove that $\hat{f}$ is infinitely differentiable at all $z > 0$.

(b) Suppose that $f$ is infinitely differentiable. Prove that $\hat{f}$ is infinitely differentiable at $z = 0$ as well, and that

$$
\hat{f}^{(n)}(0) = n! f^{(n)}(0) \text{ for all } n = 0, 1, 2, \ldots.
$$

(c) Prove that if the Taylor series of $f$ at $x = 0$ has a finite convergence radius, then the Taylor series of $\hat{f}$ has the zero convergence radius.

(d) Give an example of an infinitely differentiable function whose Taylor series has the zero convergence radius.

2. Let $P_1, P_2, P_3, \ldots$, be a sequence of non-zero polynomials on $\mathbb{R}^n$.

(a) Prove that the subset

$$
\{ x \in \mathbb{R}^n | \forall i, P_i(x) \neq 0 \}
$$

is dense, i.e. every neighborhood of every point in $\mathbb{R}^n$ contains a point where none of the polynomials vanish.

(b) For $n = 1$, give an example showing that the statement fails, when $P_i$ are not required to be polynomials, but can be arbitrary infinitely differentiable functions.

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$^1$It is closely related to Laplace’s transform.
3. Given \( M \geq 0 \) and \( L \geq 0 \), denote by \( E_{M,L} \) the set of all differentiable 2\( \pi \)-periodic functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying
\[
|f'(x)| \leq M \text{ for all } x, \text{ and } \left| \int_0^{2\pi} f(x) dx \right| \leq L.
\]

(a) Prove that every sequence \( \{f_n\} \) of functions from \( E_{M,L} \) contains a uniformly convergent subsequence.

(b) Prove that (the supremum over all functions from \( E_{M,L} \))
\[
S_{M,L} := \sup_{f \in E_{M,L}} \int_0^{2\pi} |f(x)| dx
\]
is finite (i.e. \( S < \infty \)).

(c) In the special case \( M = 1, L = 0 \), find \( S_{1,0} \).