Tautological exercises which all math majors must do once in their lifetime (according to Zassenhaus; well, more or less...)

1. Prove that in a group, a unit element is unique.

2. Prove that in a group, an inverse of every element is unique.

3. Prove that a homomorphism between two groups maps the unit element to the unit element, and maps inverse elements to inverse elements.

4. Prove that in a group, the "linear equation" $ax = b$ (resp. $ya = b$) has a unique solution.

5. Prove that in a group, cancellation rules hold: $ax = ay$ implies $x = y$, and $xb = yb$ implies $x = y$ too.

6. Prove that if a group homomorphism is bijective then the inverse map is also a homomorphism.

7. Prove that a subset in a group is a subgroup if and only if for every two elements a, b of the subset, ab^{-1} is also in the subset.

8. Prove that the range of a homomorphism is a subgroup.

9. Prove that an injective homomorphism is an isomorphism between the domain group and the range.

10. Prove that intersection of subgroups is a subgroup.

11. Prove that invertible functions from a set to itself form a group with respect to the operation of composition of functions.

12. For a subgroup $H \subset G$, prove that $x_H \equiv y$ (resp. $x \equiv_H y$ mod H) defined by $x^{-1}y \in H$ (resp. by $yx^{-1} \in H$) is an equivalence relation on G.

13. Prove that the inversion $G \to G : g \mapsto g^{-1}$ transforms left H-cosets into right H-cosets, and vice versa.

14. Prove that left (resp. right) translation by $x \in G$ establishes a bijection $H \to xH$ (resp. $H \to Hx$) between the subgroup and its left (resp. right) coset containing x, and then derive from this Lagrange's theorem: *The order of a finite group is divisible by the order of any of its subgroups.*

15. Prove that for every $g \in G$, the operation $G \to G : x \mapsto gxg^{-1}$ of conjugation by g is an *automorphism* of the group G (i.e. an isomorphism with itself).

16. Prove that the following two properties of a subgroup $H \subset G$ are equivalent: (i) left H-cosets coincide with right H-cosets (i.e. $xH = Hx$ for each $x \in G$), and (ii) H is invariant under conjugations (i.e. $gHg^{-1} = H$ for all $g \in G$).

17. Prove that kernel $Ker(f) := f^{-1}(e')$ of a group homomorphism $f : G \to G'$ is a subgroup in G, and it is normal (i.e. possesses either of the properties (i), (ii) from Problem 16).

18. Prove that (in notations of Problem 17) for every $g' \in f(G) \subset G'$, the inverse image $f^{-1}(g')$ is an H-coset.

19. Prove that if $H \subset G$ is a normal subgroup, then on the set G/H of H-cosets, there exists a unique group structure such that the projection $\pi : G \to G/H$ which associates to $x \in H$ the H-coset containing x is a group homomorphism.

20. Prove the *epimorphism theorem*: Given a group epimorphism $f : G \to G'$, there is a unique isomorphism $\hat{f} : G/Ker(f) \to G'$ such that $f = \hat{f} \circ \pi$. (Here $\pi : G \to G/Ker(f)$ is the projection of G onto the quotient group G/H — as in Problem 19 — with $H = Ker(f)$.)

21. Prove the equivalence of two definitions, (i) and (ii), of an action of a group G on a set X: (i) A homomorphism $G \to S_X$, (ii) a function $G \times X \to X : (g, x) \mapsto gx$, such that $ex = x$ for all $x \in X$, and $(g_1 g_2)x = g_1(g_2 x))$ for all $x \in X$ and all $g_1, g_2 \in G$.

22. Prove that multiplication $G \times G \to G : (g, x) \mapsto gx$, in a group G defines the action of G on itself by left translations, which is faithful and transitive, and that the mapping $G \times G \to G : (g, x) \mapsto xg^{-1}$ defines another such action — by right translations.

23. Prove Cayley's theorem: Given a group G, the map which to an element $g \in G$ associates the left translation $l_g : G \to G$ by g (defined as $l_g(x) := gx$) is an isomorphism between G and a subgroup in the group of invertible functions $G \to G$ (from the previous exercise).

24. Prove that inversion $G \to G : g \mapsto g^{-1}$ establishes an isomorphism between a group G and the *opposite* group G^{opp}, which as a set coincides with G, but is equipped with the operation defined anew, as the old operation performed in the opposite order: $x \circ_{new} y := y \circ_{old} x$.

25. Prove that replacing in Problem 22 left translations with right translations $r_g : G \to G$ (defined as $r_g(x) := xg$) results in the isomorphism between G^{opp} and a subgroup in the group of invertible functions $G \to G$.

26. Given an action of G on X, prove that "$x_1 \sim x_2$ whenever there exists $g \in G$ such that $x_2 = gx_1$" is an equivalence relation on the set X. (The equivalence classes are called the *orbits* of the action.

27. Let $Gx_0 = \{x \in X \mid x \sim x_0\}$ be the orbit containing x_0. Let $H := St(x_0) := \{g \in G \mid gx_0 = x_0\}$ be the *stabilizer* of x_0 (which is a subgroup in G). Prove that there exists a unique bijection $\phi : G/H \to Gx_0$ between the set of left H-cosets and the orbit such that the map $G \to Gx_0 : g \mapsto gx_0$ is the composition $\phi \circ \pi$, where $\pi : G \to G/H$

is the canonical projection. Show that ϕ transforms the G-action by left translations on G/H into the given action on the orbit Gx_0, i.e. $\phi(g'(gH)) = g'\phi(gH)$ for all $g, g' \in G$. (This fact is called the *orbit-stabilizer theorem.*)

28. Show that when $|G| < \infty$, we have $|Gx_0| = |G|/|St(x_0)|$.

29. Show that stabilizers of elements of the same orbit are conjugated subgroups in G.

30. Show that the map $G \times G \to G$ given by the operation of conjugation: $(g, x) \mapsto gxg^{-1}$ defines an action of G on itself.

31. Show that the action of G on itself by conjugations defines a homomorphism $G \to Aut(G) \subset S_X$ from G to the group of all automorphisms of G (which is a subgroup in the group of permutations on the set G). [By definition, the range of this homomorphism consist of *interior automorphisms.*]

32. Prove that the kernel of the homomorphism $G \to Aut(G)$ (defined in the previous problem) is the *center* Z of G (by definition, it consists of all those elements of G which commute with all elements of G: $Z = \{g \in G \mid gx = xg \ \forall x \in G\}$.).

33. Prove that interior automorphisms form a normal subgroup in the group $Aut(G)$ of all automorphisms.