1. Express $\det(\text{adj}(A))$ in terms of $\det A$, where $A$ is an $n \times n$-matrix.

Since $A \text{adj}(A) = (\det A)I$, we have: $(\det A) \det(\text{adj}(A)) = (\det A)^n$. Thus, when $\det A \neq 0$, $\det(\text{adj}(A)) = (\det A)^{n-1}$. The same remains true when $\det A = 0$. This is obvious when $A = 0$ since then $\text{adj}(A) = 0$. When $A \neq 0$ but $\det A = 0$, we still have $A \text{adj}(A) = 0$, which is possible only if $\text{adj}(A)$ is degenerate.

2. Solve system of linear equations:

$$
\begin{align*}
    x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\
    x_2 - x_3 + x_4 &= -3 \\
    x_1 + 3x_2 - 3x_4 &= 1 \\
    -7x_2 + 3x_3 + x_4 &= -3
\end{align*}
$$

Following the row reduction algorithm, we subtract the 1st equation from the 3rd one, then subtract the 2nd equation 5 times from the 3rd one, add it 7 times to the 4th one, and finally add what has become the 3rd equation twice to the 4th one. We arrive at the following system:

$$
\begin{align*}
    x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\
    x_2 - x_3 + x_4 &= -3 \\
    2x_3 - 4x_4 &= 12 \\
    0 &= 0
\end{align*}
$$

Denoting by $t$ the value of $x_4$, which can be arbitrary, and performing the back substitution, we find:

$$
x_4 = t, \quad x_3 = 6 + 2t, \quad x_2 = 3 + t, \quad x_1 = -8.
$$

3. Use Sylvester’s rule to find inertia indices of quadratic form:

$$x_1x_2 - x_2^2 + 2x_2x_4 + x_4^2.$$
It is convenient, before applying Sylvester’s rule, to reorder the variables: 
\( x_1 = y_4, \ x_2 = y_3, \ x_3 = y_2, \ x_4 = y_1 \). Then, in the y-coordinates, the quadratic form has coefficient matrix:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & -1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0
\end{bmatrix}.
\]

Computing the leading minors, we find:

\[
\Delta_0 := 1, \ \Delta_1 = 1, \ \Delta_2 = 1, \ \Delta_3 = -2, \ \Delta_4 = -1/4.
\]

In the sequence +++, +, −, −, of their signs, there is one sign change, and therefore the negative and positive inertia indices of this quadratic form are 
\( q = 1 \) and \( p = 4 - q = 3 \).

4. Transform quadratic form \( x_1x_2 + x_3x_4 \) to the normal form by an orthogonal transformation.

We apply 45°-rotations of the coordinate system in each \((x_1, x_2)\)-plane and in \((x_3, x_4)\)-plane:

\[
x_1 = \frac{y_1 - y_2}{\sqrt{2}}, \ x_2 = \frac{y_1 + y_2}{\sqrt{2}}, \ x_3 = \frac{y_3 - y_4}{\sqrt{2}}, \ x_4 = \frac{y_3 + y_4}{\sqrt{2}}.
\]

In the new coordinates, the quadratic form is \( y_1^2/2 - y_2^2/2 + y_3^2/2 - y_4^2/2 \). Switching \( y_2 \) and \( y_3 \) we obtain the standard normal form of the sum of the squares of coordinates with coefficients 1/2, 1/2, −1/2, −1/2 ordered non-increasingly.

5. Find the Jordan normal form of matrix:

\[
\begin{bmatrix}
0 & 3 & 3 \\
-1 & 8 & 6 \\
2 & -14 & -10
\end{bmatrix}.
\]

Denote the matrix by \( A \). First, we compute the characteristic polynomial \( \det(\lambda I - A) \), and find: \( \lambda^3 + 2\lambda^2 + \lambda = \lambda(\lambda + 1)^2 \). Therefore \( A \) has the eigenspace of dimension 1 corresponding to the eigenvalue \( \lambda = 0 \), and the root space of
dimension 2 corresponding to $\lambda = -1$. Next, we look for the dimension of the eigenspace corresponding to eigenvalue $\lambda = -1$. For this, we examine the matrix $A + I$:

$$
\begin{bmatrix}
1 & 3 & 3 \\
-1 & 9 & 6 \\
2 & -14 & -9
\end{bmatrix}.
$$

It has rank 2 (since $\begin{vmatrix} 1 & 3 \\ -1 & 9 \end{vmatrix} = 12 \neq 0$). Therefore the eigenspace corresponding to $\lambda = -1$ has dimension 1 while the root space has dimension 2. Thus the Jordan normal form of the matrix is:

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 1 \\
0 & 0 & -1
\end{bmatrix}.
$$

6. Can a non-zero anti-symmetric matrix be nilpotent? If “yes” give an example, if “no” provide a proof.

**Answer:** No.

**Proof:** According to the Spectral Theorem, every anti-symmetric matrix is diagonalizable over $\mathbb{C}$. To be nilpotent, a diagonal matrix must be zero, in which case a matrix similar to it is zero too.

7. Classify all linear operators in $\mathbb{R}^2$ up to linear changes of coordinates.

When the characteristic polynomial has complex conjugate roots $a \pm bi$ (where we may assume $b > 0$), the matrix is similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. When the characteristic polynomial has two distinct real roots $\lambda_1 > \lambda_2$, the matrix is similar to the diagonal one: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. In the remaining case when the characteristic polynomial has double root $\lambda_0$, the matrix is similar to scalar matrix $\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$ or to Jordan cell $\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$ depending on whether the eigenspace has dimension 2 or 1.
8. Find all those values of $a_1, \ldots, a_n$ for which the following matrix is nilpotent:

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 1 \\
-a_n & -a_{n-1} & \cdots & -a_2 & -a_1
\end{bmatrix}
$$

Using the cofactor expansion with respect to the 1st column and applying induction on $n$, it is easy to show that the characteristic polynomial of the matrix is equal to $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$. As it follows from the Jordan Canonical Form Theorem, a matrix is nilpotent if and only if all complex roots of the characteristic polynomial are equal to zero. Therefore our matrix is nilpotent if and only if $a_1 = \cdots = a_n = 0$.

9. Find out if the following quadratic hypersurfaces in $\mathbb{C}^4$ can be transformed into each other by linear homogeneous changes of coordinates:

$$
\begin{align*}
z_1z_2 + z_2z_3 + z_3z_4 &= 1 \\
z_1^2 + z_2^2 + z_3^2 + z_4^2 &= z_1 + z_2 + z_3 + z_4.
\end{align*}
$$

Quadratic form $z_1z_2 + z_2z_3 + z_3z_4$ from the first equation has coefficient matrix

$$
\frac{1}{2}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
$$

The determinant of this matrix (as it is easy to see) is non-zero, and hence the quadratic form is equivalent over $\mathbb{C}$ to $z_1^2 + z_2^2 + z_3^2 + z_4^2$.

Completing squares in the second equation, we transform it to the form

$$
(z_1 - \frac{1}{2})^2 + (z_2 - \frac{1}{2})^2 + (z_3 - \frac{1}{2})^2 + (z_4 - \frac{1}{2})^2 = 1.
$$

Thus, both hypersurfaces are equivalent to the complex sphere

$$
z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1.
$$
10. Prove that any orthogonal transformation in $\mathbb{R}^4$ with the determinant equal to $-1$ has an invariant 3-dimensional subspace.

As it follows from the real version of the Spectral Theorem applied to orthogonal transformations, each orthogonal transformation in $\mathbb{R}^4$ with determinant $-1$ in a suitable orthonormal basis is described by matrix:

$$
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
$$

In particular, it has real eigenvectors with the eigenvalues $-1$ (the last column) and $1$ (next to last). The 3-dimensional subspace perpendicular to either of these eigenvectors is invariant.