Math 215A. Fall 2019. Final exam. Solutions

1. Give a homotopy theory proof of the following algebraic theorem: A subgroup of a free group is a free group.

`Solution.` The bouquet of circles labeled by the generators of a free group F has F as its fundamental group. (In fact, it is the Eilenberg-MacLane space $K(F,1)$.) The universal cover of the bouquet is an infinite graph — a connected tree, or 1-dimensional CW-complex — on which a given subgroup $H \subset F$ acts by deck transformations (in fact by automorphisms of the graph). The quotient of the tree by H is still a connected graph which has H as its fundamental group (and is in fact $K(H,1)$). By contracting a maximal tree of it (here the axiom of choice is used to guarantee its existence, and the Borsuk property of CW-pairs is used to guarantee that the contraction does not change the homotopy type) we obtain a bouquet of circles whose fundamental group H is therefore free.

`Remark.` Many exam solutions resembled too much the arguments from *Hatcher*, and unfortunately not everyone acknowledged the influence.

2. Prove that each skeleton $sk_n X$ of a contractible CW-complex X is homotopy equivalent to a bouquet of n-spheres.

`Solution.` For $n = 0$, the skeleton is a discrete set and hence a bouquet of 0-spheres. For $n = 1$, it is a graph, connected since $\pi_0(sk_1 X) = \pi_0(X)$, and hence homotopy equivalent to a bouquet of circles as in the argument above. Since in general $\pi_k(X)$ is determined by the $(k+1)$-skeleton of X, we conclude that $sk_n X$ is $(n-1)$-connected. For $n > 1$, Hurewicz' theorem implies that $H_k(sk_n X) = 0$ for all $k < n$, and $\pi_n(sk_n X) = H_n(sk_n X)$. The latter is a free abelian group, since for an n-dimensional cell space is, it the kernel of the boundary operator in the cellular chain complex, and hence a subgroup in the free abelian group of chains. Now take a bouquet Y_n on n-spheres labeled by a set of free generators of $\pi_n(sk_n X)$, and map it to $sk_n X$ by the bouquet of spheroid maps realizing the generators. This map induces isomorphism in homology, and hence is a homotopy equivalence by Whitehead's homological theorem.

`Remark.` Some attempted solutions neglecting to use Hurewicz' and Whitehead's theorems argued (correctly) that $sk_{n-1}X$ is contractible in $sk_n X$ to conclude (incorrectly) that $sk_n X$ is homotopy equivalent to the bouquet of n-spheres $sk_n X / sk_{n-1} X$. To see what's wrong it suffices to look at the example of, say, disk D^2, which is not homotopy equivalent to $D^2/\partial D^2 = S^2$.

3. Prove that $\Omega G_+(\infty, n)$ is weakly homotopy equivalent to SO_n.

`Solution 1.` It follows from homotopy sequences of the fibrations $EG \stackrel{G}{\to} BG$ and $P(BG) \stackrel{\Omega BG}{\to} BG$ (where $P(X)$ stands for the space of based paths in (X, x_0), and is contractible, as well as EG is) that $\pi_k(G) = \pi_{k+1}(BG) = \pi_k(\Omega G)$ for all k. To guarantee the WHE, we need a map between the spaces that would induce the isomorphisms.

A map $\Omega BG \to G$ would require a canonical lifting of loops; in terms of the bundle $V(\infty, n) \to G_+(\infty, n)$, picking an orthonormal frame for every loop of oriented n-dimensional subspaces in $\mathbb{R}^\infty$. There is no a natural way of doing this. However, on can introduce and use a *connection* (a "gauge field" in physicists' terms). To realize this plan, one shows first that smooth loops form a subspace $\Omega_{sm} G_+(\infty, n)$ weakly homotopy equivalent to $\Omega G_+(\infty, n)$. Then, using a Riemannian metric on the Stiefel manifold, one takes at each point of it the tangent subspace orthogonal to the fiber (i.e. to the orbit of SO_n). The usual notion of connection requires such a field of tangent orthogonal complements to the fibers to be SO_n-invariant, which can be achieved by first taking the Riemannian metric invariant (by averaging any seed metric over the action of the compact group). In any case, once such orthogonal complements are selected, smooth paths from the base can be canonically lifted (by solving appropriate ODEs) to paths in the Stiefel manifold starting an a base point (an initial frame) over the base point in the grassmannian. Assigning to a path from $P_{sm}(G_+(\infty, n))$ the endpoint of its lift, we obtain a map $P_{sm}(G_+(\infty, n)) \to V(\infty, n)$ which commutes with the projections to the base, and thus induces the needed maps between the homotopy sequences, and (by the five-lemma) a WHE $\Omega_{sm} G_+ \infty, n) \to SO_n$ between the fibers.

`Solution 2.` It is actually easier to construct a map in the opposite direction: $G \to \Omega BG$. Namely, for a finite-dimensional Lie group G and cellular BG, one may assume EG also cellular (in any case, our Stiefel manifold $V(\infty, n)$ is an inductive limit of compact manifolds, and this is all that's needed), and so EG is contractible. Let $\Phi : EG \times [0, 1] \to EG$ be a homotopy between the constant map $EG \to x_0 \in EG$ and the identity. Composing Φ with the projection $\pi : EG \to BG$, we obtain a map from EG to the space of paths in BG starting at $y_0 = \pi(x_0)$. This map commutes with the projections of both spaces to BG, and hence provides the needed morphism between the homotopy sequences, and a WHE between the fibers.

`Solution 3` (inspired by one exam paper). Put $M = \{(x,p) \in EG \times P(BG, y_0) \mid p(1) = \pi(x)\}$. Projection $(x,p) \mapsto p$ is a locally trivial fibration of M over the (contractible!) path space with the fiber $\pi^{-1}(p(1)) \cong G$. Thus, the inclusion $G \subset M$ of the fiber into the total space is a WHE. Projection $(x,p) \mapsto x$ is a Hurewicz fibration over (weakly contractible!) EG with the fiber ΩBG (over any $x_0 \in \pi^{-1}(y_0)$). Thus, the inclusion $\Omega BG \subset M$ of the fiber into the total space is a WHE.

`Solution 4` (inspired by another exam paper). For any cell space X, $\pi(X, BG) = St(X; G)$, the set of equivalence classes of principal G-bundles over X. Likewise, $\pi(X, \Omega BG) = \pi(\Sigma X^+, BG) = St(\Sigma X^+; G)$.

There are some subtleties here. First, BG has no natural base point, so one needs to pick one. Next, since X comes without base point, the first equality requires introducing $X^+ = X \sqcup pt$, so that the suspension ΣX^+ is in fact a Thom space — one-point compactification of $X \times \mathbb{R}$. The second equality requires therefore the base-point version of the bundle classification theory, where one may assume that the fiber over the base point is identified with G once and forever.

Now, such a principal G-bundle over the suspension ΣX^+, which is glued from two cones $C_\pm X$ over their bases and at the vertices, can be trivialized over the cones $C_\pm X$ and then described by the comparison map $g : X \mapsto G$ over the glued bases. Thus, $St(X; G)$ can be described as the quotient of the group G^X of maps $X \to G$ by the action of the group of re-trivializations $\psi_\pm : C_\pm X \to G$, which must be standard at the vertices: $\psi_\pm(v_\pm) = e$, and act via the left and right translations: $g \mapsto \psi_+|_X \; g \; \psi_-^{-1}|_X$. The group of re-trivializations satisfying $\psi_\pm(v_\pm) = e$ is contractible (together with the cones themselves), so we obtain a (surjective) map $St(\Sigma X^+; G) \to \pi(X, G)$ to the set of path-connected components of G^X. Injectivity is also clear: if $G : X \times [0,1] \to G$ is a homotopy between g_0 and g_1, then $G(x,t)g_0(x)^{-1}$ factors through the map $\psi : (CX, v) \to (G, e)$, and $g_1 = \psi|_{t=1} g_0$. Thus, we obtain a bijection between $\pi(X, \Omega BG)$ and $\pi(X, G)$, which is (as it is easy to see) natural in X. Therefore ΩBG and G are weakly homotopy equivalent.

4. Prove that S^2 smoothly embedded (or even immersed) in $\mathbb{C}^2$ has at least two distinct points where the tangent planes are complex lines.

`Solution.` In fact replacing “embedded” (as it is stated in our text) with “immersed”, I am afraid, I was overly optimistic (and I suspect, counterexamples exist). So, my solution does use the fact that the surface is embedded in $\mathbb{R}^4$.

Namely, both the tangent and normal bundles are orientable, and are induced by the *Gauss map* $g : S^2 \ni x \mapsto T_x S^2 \in G_+(4,2)$ from the tautological bundle over the grassmannian and from its complementary bundle respectively. The self-intersection index of the zero section in the tangent bundle is equal to the Euler characteristic χ (as for any closed oriented manifold), i.e. 2 for the sphere. The similar self-intersection index for the normal bundle is 0. This is because it coincides with the self-intersection index of the surface in $\mathbb{R}^4$, which is acyclic (or, simply speaking, since one can translate the surface far away from itself). Here is where the answer for immersions would be different: each transverse self-intersection point of the immersed surface would add ± 2 to the self-intersection index in the normal bundle, with the zero total as the self-intersection index in the ambient $\mathbb{R}^4$.

On the other hand, $G_+(4,2) = S^2 \times S^2$ has its 2nd homology group isomorphic to $\mathbb{Z}^2$. The idea now is that the two intersection numbers, $\chi = 2$ and 0, determine the homology class $g_*[S^2] \in \mathbb{Z}^2$. Then the intersection index of this class with $[\mathbb{C}P^1] \subset G_+(4,2)$ formed by complex lines in $\mathbb{C}^2 = \mathbb{R}^4$ can be found from any example. E.g. take the standard unit sphere in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \subset \mathbb{C}^2$ to find that it has two tangent planes parallel to $\mathbb{C} \times 0$, of which just one is oriented as the complex line. So, the intersection index equals 1, implying that every embedded sphere must have a point where the *oriented* tangent plane is a complex line with the complex orientation. The same result for the $\mathbb{C}P^1$ of complex lines with the anti-complex orientation (they form a locus in $G_+(4,2)$ disjoint from the previous $\mathbb{C}P^1$) guarantees the existence of a 2nd tangent plane parallel to a complex line.

To show that the two intersection numbers determine the Gauss class $g_*[S^2]$, one only needs to check that the 1-st Chern classes of the two $SO(2) = U(1)$-bundles over $G_+(4,2)$ — the tautological one, and its complementary — form a coordinate system on $H_2(G_+(4,2))$. This can be approached by direct computation, which may look rather arbitrary.

In fact the situation is somewhat subtle. One could think that the 1st Chern classes of the two complementary $SO(2)$-bundles should be opposite (and hence linearly dependent). However, the sum of the bundles is trivial as an $SO(4)$-bundle, not as a $U(2)$-bundle, and as a result the expectation is incorrect. (According to Michael Atiyah, this is what physicists call an *anomaly*: you solve a problem, and discover that your expectations were wrong!) Instead, there are two inclusions of $S^2 \times S^2 = G_+(4,2)$ into $G_+(\infty,2)$ (which, as we know, is homotopy equivalent to $\mathbb{C}P^\infty$), each inducing one of the two tautological $SO(2)$-bundles. In terms of Plücker coordinates P_{ij},

$1 \le i < j \le 4$ on the grassmannian $P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$, $\sum P_{ij}^2 = 1$, the involution interchanging an oriented plane in $\mathbb{R}^4$ with its orthogonal complement, acts as the Hodge $*$-operator: $P_{12} \leftrightarrow P_{34}$, $P_{13} \leftrightarrow -P_{24}$, $P_{14} \leftrightarrow P_{23}$. In terms of the product $S^2 \times S^2$, this operation acts as the central symmetry on one of the factors (and identity on the other). In the coordinates (x, y) on $H_2(S^2 \times S^2) = \mathbb{Z}^2$, let the involution be $(x, y) \mapsto (x, -y)$, let $ax + by$ be the 1st Chern class of the tautological bundle, and hence $ax - by$ be the 1st Chern class of the complementary one. For the aforementioned standard embedding of S^2 in $\mathbb{R}^4$, the Gauss map lands in a plane $P_{34} = P_{24} = P_{23} = 0$ (in suitable Plücker coordinates). In terms of $S^2 \times S^2$, this is a diagonal sphere (or anti-diagonal, depending on the choice of orientations of the factors). So, we may assume that $g_*[S^2] = (1, 1)$ in our coordinates. Then the values $a + b = 2$ and $a - b = 0$ show that $a = b = 1$, i.e. the two 1st Chern classes have the form $x + y$ and $x - y$ and are indeed linearly independent.

Well, maybe there is a shorter way of checking this.

5. Let f be a continuous map f from the n-torus $\mathbb{R}^n/\mathbb{Z}^n$ to itself, and let $A : \mathbb{Z}^n \to \mathbb{Z}^n$ denote the homomorphism induced by f on the fundamental group of the torus. Prove that f has at least $|\det(I - A)|$ *distinct* fixed points.

`Solution.` The homology $H_\bullet(T^n)$ of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the "exterior algebra" $\Lambda^\bullet(\mathbb{Z}^n)$ of the lattice. Consequently, $\det(I - A)$, as it was calculated in some exam papers, is the super-trace of f_* on $H_\bullet(T^n)$. Thus, $\det(I - A)$ is the Lefschetz number of f. When it is non-zero, the linear map $x \mapsto Ax : T^n \to T^n$ has $|\det(I - A)|$ non-degenerate fixed points each contributing $\pm 1 = \operatorname{sign}\det(I - A)$ into the Lefschetz formula. Thus, A has as few fixed points as the map f having only non-degenerate fixed points can possibly have in agreement with Lefschetz' lower bound. Yet, this is not what the problem is about, since it asks to show that even when fixed points are allowed to degenerate, their number cannot become smaller than this bound.

Intuitively this suggests that somehow the fixed points of A are all "topologically different" in nature, and when A is deformed into a (non-linear) f, new fixed points can be born or die (usually in pairs), but there should remain at least one such fixed point of each of $|\det(I - A)|$ topological types.

One way to make sense of this intuition would be to start with the linear map $I - A : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$. It is a covering of degree $\det(I - A)$ (assuming that the number is non-zero), and maps $\mathbb{Z}^n = \pi_1(T^n)$ to its own sublattice of index $|\det(I-A)|$. Since T^n is a group, we can consider the difference $\mathrm{id}-f$ even when f is non-linear (think of mapping T^n to the graph of f in $T^n \times T^n$, and then projecting along the diagonal). The hypothesis that $A = f_*$ implies that $(\mathrm{id}-f)_* = I - A$, i.e. that the image $(\mathrm{id}-f)_*\pi_1(T^n)$ is the same sublattice of index $|\det(I-A)|$ in $\pi_1(T^n)$. Now the homotopy lifting property of coverings shows that $\mathrm{id}-f = (I-A)\circ g$, i.e. factors through a degree-1 map $g : T^n \to T^n$. The fixed points of f are zeroes of $\mathrm{id}-f$, i.e. inverse images under g of the fixed points of A. Since g, being a non-zero degree map, is surjective, the result follows.

6. In a closed manifold M, let $Z_1, \ldots, Z_n$ be closed submanifolds of positive codimensions such that the cup-product of the Poincaré-duals of their fundamental classes in non-zero in $H^\bullet(M;\mathbb{Z}_2)$: $D^{-1}[Z_1]\cup\cdots\cup D^{-1}[Z_n] \neq 0$. Prove that a smooth function on M has at least $n+1$ distinct critical points, and conclude that this lower bound holds for functions on $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$.

`Solution.` The solutions of this problem submitted in the exam were from those who had mastered the Lusternik-Schnirelmann theory beforehand, and knew that the number of geometrically distinct critical points of a function on a manifold is bounded below by the so-called *category*, and that the latter one is bounded below by the so-called *cup-length*. This approach brings to mind the joke about a mathematician who, instead of boiling a water-filled tea-kettle by putting it on the stove (as every physicist would do), pours the water out, thereby reducing the problem to the previous, "already solved" one, when the given tea-kettle was empty.

Here is a straightforward, "physicist's" solution. Given a smooth function $f : M \to R$, suppose we have a cycle (e.g. the fundamental cycle $[C]$ of some submanifold) sitting in the part $M_+ = \{x \in M \mid f(x) \leq f_{crit} + \epsilon\}$ of the manifold beneath a level of f slightly above a critical one. We can try to use a Riemannian metric and the flow of the gradient vector field $\dot{x} = -\nabla f(x)$ in order to pull the cycle to the part $M_- \subset M_+$ beneath a level $f_{crit} - \epsilon$ slightly below the critical one. We will succeed, provided that C is disjoint from ϵ-size neighborhoods of the critical points of f with the critical value f_{crit}, since outside such neighborhoods the gradient flow contracts M_+ to M_-. However, even if we fail, i.e. in the process of deformation C "gets stuck" at the critical points, we can take the cap-product $C \cap D^{-1}[Z]$ of C with a class Poincaré-dual to a positive codimension cycle. We can assume that such a cycle avoids those dangerous ϵ-neighborhoods of the critical points, since the critical point are isolated (if not, they are infinitely many, so there is nothing to prove). To avoid technical complications with Peano maps, triangulations, or smooth approximations of singular cycles, we assume that Z is a submanifold. Since the cap-product is supported near Z, it is represented (geometrically by the transverse intersection $[C \cap Z]$) by a cycle which can be pulled down to M_- without obstructions.

Now, imagine that the total number of critical points of f is finite, but the number of critcal levels $\leq n$. Then, starting with $[M]$, and replacing it inductively with $[M] \cap D^{-1}[Z_1]$, then $[M] \cap D^{-1}[Z_1] \cap D^{-1}[Z_2]$, etc. every time we cross a critical level, we end up after n steps with a nonzero class $D^{-1}[Z_1] \cup \cdots \cup D^{-1}[Z_n]$ Poincaré-dual to a cycle in the empty set $\{x \in M \mid f(x) < \min f\}$ — a contradiction!

7. Prove that $H^\bullet(\mathbb{C}G(4,2))$ can be described as the ring generated by classes c_1 and c_2 of degrees 2 and 4 respectively, which satisfy the relation $(1 + c_1 + c_2)(1 + c_1' + c_2') = 1$. More precisely, this identity allows one to express classes c_1' and c_2' of degrees 2 and 4 via c_1 and c_2, and in addition to provide a complete set of relations between c_1 and c_2.

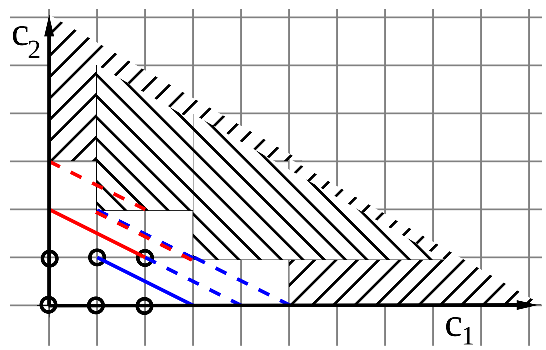
`Solution.` The trivial bundle $\mathbb{C}^4$ over the grassmannian is topologically the direct sum of the two 2-dimensional bundles: the tautological one and its complementary. The relation represents, of course, the total Chern class 1 of the trivial bundle as the product of the total Chern classes of the two summands. Two things are left to show: (i) that classes c_1, c_2 generate the whole ring, and (ii) that all relations follow from the given one.

For (i), note that the standard inclusion $\mathbb{C}G(4,2) \subset \mathbb{C}G(\infty,2)$ (it induces the tautological $\mathbb{C}^2$-bundle from the universal one) induces an epimorphism in cohomology (because it is injective at the level of Schubert cells, and hence in homology, which is freely generated by the cells). Since $H^\bullet(\mathbb{C}G(\infty,2))$ is the polynomial ring generated by the universal Chern classes, we conclude that indeed $H^\bullet(\mathbb{C}G(4,2))$ is generated as a ring by the Chern classes c_1, c_2 of the tautological bundle.

It is likewise generated by the Chern classes c_1', c_2' of the complementary tautological bundle, and as we will see, the above relation allows one to re-express c_i' via c_j and *vice versa*. So far everything carries *verbatim* to any $\mathbb{C}G(m+n,n)$. However, in the absence of any algebraic (or topological) machinery, it would be challenging to establish part (ii) in general.

Componentwise, the relation reads: $c_1 + c_1' = 0$, $c_2 + c_2' + c_1 c_1' = 0$, $c_1 c_2' + c_2 c_1' = 0$, and $c_2 c_2' = 0$. The first two yield $c_1' = -c_1$ and $c_2' = c_1^2 - c_2$ (as promised). Eliminating the primes from the last two, we get: $c_1^3 - 2c_1c_2 = 0$ and $c_2^2 - c_1^2c_2 = 0$. Form this, it is not hard to see that in degrees 0,2,4,6,8, the monomials 1, c_1, c_1^2 and c_2, c_1c_2, $c_1^2c_2$ form a $\mathbb{Z}$-basis, while all other monomials of the same degrees (namely c_1^3, c_1^4 and $c_1^2c_2$) are their linear combinations due to the above relations. Thus the dimensions of the graded components of the ring $\mathbb{Z}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_2^2 - c_1^2c_2)$ in these degrees are $1, 1, 2, 1, 1$ just as the corresponding numbers of the Schubert cells of $\mathbb{C}G(4, 2)$.

All the papers I've got stopped here, but didn't show that this quotient ring has no components in higher degrees. This is not hard, but can be messy — unless one resorts to the pictorial technique of *Newton diagrams*.



The grid's axes labeled c_1 and c_2 represent not the values of the letters, but the exponents of monomials $c_1^m c_2^n$. The solid colored lines connect the monomials participating in the generators $c_1^3 - 2c_1c_2$ (blue) and $c_2^2 - c_1^2c_2$ (red) of the ideal. The dashed lines are the consequences of these relations obtained by multiplying the same-color relations by c_1 and c_2 (red) and c_1, c_1^2, and c_2 (blue). Note that $c_2(c_1^3 - 2c_1c_2) = 0$ and $c_1(c_2^2 - c_1^2c_2) = 0$ (shown as coinciding blue and red lines) are linearly independent (due to the coefficient 2 lost in our pictorial notation), and hence imply that both participating monomials, $c_1c_2^2$ and $c_1^2c_2$, are in the ideal. Once a monomial is in the ideal, all its products with other monomials $c_1^m c_2^n$ are also in the ideal. This is shown by shading the whole quadrants with the corners at our two monomials. Now, each relation line with one vertex in the shaded area has therefore the other vertex in the ideal two. This way we can shade the quadrants with vertices c_1^5 and c_2^3. This shades the entire diagram except a few remaining monomials. The circles show which ones can be taken for a $\mathbb{Z}$-basis, while the lines show how to express the uncircled monomials via the circled ones. It remains to mention that the numbers of the circles and Schubert cells match.