Math 215A. Fall 2019. Final exam

Solve 4 out of the following 7 problems, and submit your solutions by Tuesday, December 17. (You may submit more than 4, but only the best 4 will count.)

1. Give a homotopy theory proof of the following algebraic theorem: A subgroup of a free group is a free group.

2. Prove that each skeleton $sk_n X$ of a contractible CW-complex X is homotopy equivalent to a bouquet of n-spheres.

3. Prove that $\Omega G_+(\infty, n)$ is weakly homotopy equivalent to SO_n.

4. Prove that S^2 smoothly embedded (or even immersed) in $\mathbb{C}^2$ has at least two distinct points where the tangent planes are complex lines.

5. Let f be a continuous map f from the n-torus $\mathbb{R}^n/\mathbb{Z}^n$ to itself, and let $A : \mathbb{Z}^n \to \mathbb{Z}^n$ denote the homomorphism induced by f on the fundamental group of the torus. Prove that f has at least $|\det(I - A)|$ *distinct* fixed points.

6. In a closed manifold M, let $Z_1, \dots, Z_n$ be closed submanifolds of positive codimensions such that the cup-product of the Poincaré-duals of their fundamental classes in non-zero in $H^\bullet(M; \mathbb{Z}_2)$: $D^{-1}[Z_1] \cup \dots \cup D^{-1}[Z_n] \neq 0$. Prove that a smooth function on M has at least $n + 1$ distinct critical points, and conclude that this lower bound holds for functions on $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$.

7. Prove that $H^\bullet(\mathbb{C}G(4, 2))$ can be described as the ring generated by classes c_1 and c_2 of degrees 2 and 4 respectively, which satisfy the relation $(1 + c_1 + c_2)(1 + c_1' + c_2') = 1$. More precisely, this identity allows one to express classes c_1' and c_2' of degrees 2 and 4 via c_1 and c_2, and in addition to provide a complete set of relations between c_1 and c_2.