Math H110. Honors Linear Algebra
Final Exam, 12.16.15

Out of the following 12 problem, choose 6 and solve. You may solve more than 6 problems, but only best 6 results will be counted.

1. Prove that on the Euclidean plane, every linear transformation with positive determinant is the composition of a rotation with the expansion (or compression) in two perpendicular directions with (possibly different) positive coefficients.

`Solution.` This is an instance of "polar factorization". A linear transformation A maps the unit circle into an ellipse, which can be transformed back into the unit circle by a linear transformation S compressing the ellipse in two perpendicular directions (its principal axes). Thus $U = SA$ preserves the unit circle and has determinant $+1$, if $\det A > 0$. Thus U is a rotation, and $A = S^{-1}U$.

2. Compute e^{tA} where $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$.

`Solution.` A is the realification of the multiplication by $2 + 3i$ on $\mathbb{C}$, and hence e^{tA} is the realification of $e^{(2+3i)t}$:

$$e^{tA} = e^{2t} \begin{bmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{bmatrix}.$$

3. Prove that every invertible complex matrix can be represented as the product of a unitary matrix and an upper-triangular matrix.

`Solution.` This is the matrix form of the Gram-Schmidt orthogonalization. Columns of an invertible matrix A represent a basis in $\mathbb{C}^n$. The orthogonalization with respect to the standard Hermitian inner product transforms the basis into an orthonormal basis inscribed into the standard flag.of columns of a unitary matrix U. Thus $A = UC$, where C is the transition matrix from the orthonormal basis to the original. Since C preserves the standard flag, it is upper-triangular.

4. Prove that every orthogonal transformation has an orthogonal cubic root.

`Solution.` By the real version of the spectral theorem, for any orthogonal transformation U, the space can be represented as a direct orthogonal sum of U-invariant planes on which U acts as rotation through some angle θ, and invariant lines on which U acts by ± 1.

Replacing each rotation through the angle θ by its cubic root, the rotation through $\theta/3$, and taking in account that ± 1 are cubic roots of themselves, we obtain an orthogonal cubic root of U.

5. Let graph Γ have the shape of letter T with the three legs consisting of $p-1$, $q-1$, and $r-1$ edges respectively (and hence with $n = p+q+r-2$ vertices totally). Consider the quadratic form

$$Q_\Gamma(x_1,\dots,x_n) := 2 \sum_{\text{vertices } v_i} x_i^2 - 2 \sum_{\text{edges } e_{ij}} x_i x_j.$$

Compute the inertia indices of Q_Γ assuming that $1/p + 1/q + 1/r < 1$.

`Solution.` Erasing the common vertex (let us call it the first one) of the three legs of the graph together with the adjacent three edges, we obtain the graph with three connected components $A_{p-1}, A_{q-1}, A_{r-1}$ (in the notation of section "Quivers"), and cutting further the vertices adjacent to v_1 leaves the graphs $A_{p-2}, A_{q-2}, A_{r-2}$ respectively. As it was found in section "Quivers" by induction on n and cofactor expansion, $\det Q_{A_n} = n+1$. This implies (e.g. by Sylvester's rule) that on the subspace $x_1 = 0$ quadratic form Q_Γ is positive definite, and hence the positive inertia index is at least $n-1$. In fact it is $n-1$, while the negative inertia index is 1. Indeed, using the cofactor expansion with respect to the first row and column of the matrix Q_Γ, we find

$$\begin{aligned}
\det Q_\Gamma &= 2 \det Q_{A_{p-1}} \det Q_{A_{q-1}} \det Q_{A_{r-1}} - \det Q_{A_{p-2}} \det Q_{A_{q-1}} \det Q_{A_{r-1}} \\
&\quad - \det Q_{A_{p-1}} \det Q_{A_{q-2}} \det Q_{A_{r-1}} - \det Q_{A_{p-1}} \det Q_{A_{q-1}} \det Q_{A_{r-2}} \\
&= 2pqr - (p-1)qr - p(q-1)r - pq(r-1) \\
&= pqr\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right) < 0.
\end{aligned}$$

6. Let graph Γ have the shape of a hexagon. Find all critical values of the quadratic form Q_Γ (defined in Problem 5) on the unit sphere $x_1^2 + \cdots + x_6^2 = 1$.

`Solution.` According to the Orthogonal Diagonalization Theorem, in a suitable coordinate system $Q_\Gamma(x) = \sum_i \lambda_i y_i^2$ and $\sum_i x_i^2 = \sum_i y_i^2$. The critical values of Q_Γ under the constraint $\sum_i x_1^2 = 1$ are, as it is easy to see e.g. from the Lagrange multiplier method, are λ_i, the spectral numbers of the pair of the quadratic forms or, equivalently, the eigenvalues of the corresponding symmetric operator.

The matrix of $Q_\Gamma(x)$ equals $2I - T - T^{-1}$, where T is the matrix of the cyclic shift of coordinates: $T(x_1,\dots,x_6) = (x_2,\dots,x_6,x_1)$. The complex eigenvectors of T form the "Fourier basis", and the eigenvalues are the 6th roots of unity: $\pm 1, 1/2 \pm i\sqrt{3}/2, -1/2 \pm i\sqrt{3}/2$. The eigenvalues of $2-T-T^{-1}$ are therefore $0,4,1,1,3,3$. Thus, the critical values are $0,1,3,4$.

7. Prove that real anti-symmetric matrices have even rank and non-negative determinant.

`Solution.` According to real version of the Spectral Theorem, an antisymmetric $n\times n$-matrix A can be transformed by operations $A\mapsto C^tAC$ (where C is orthogonal) to the block-diagonal form with the *non-zero* blocks having the form 2×2-blocks $\begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}$. Thus, the rank of A is *twice* the number s of such 2×2-blocks, i.e. even, the determinant is non-zero only when $s=m/2$, in which case $\det A = (\det C)^2\prod_{i=1}^s \omega_i^2 > 0$.

8. Prove that any constant coefficient differential operator

$$D := (d/dt)^n + a_1\,(d/dt)^{n-1} + \cdots + \cdots a_n\,(d/dt)^0$$

maps the space of all polynomials in t onto itself, i.e. *surjectively*.

`Solution.` Let r be the *smallest* index k such that $a_k\neq 0$. Then the null-space of D on the space of polynomials coincides with the space $\mathcal{P}_r$ of dimension r consisting of all polynomials of degree $<r$. On the other hand, such D lowers the degree of any polynomial by r, and therefore maps $\mathcal{P}_{N+r}$ to $\mathcal{P}_N$. Since the rank equals the dimension of the domain space less the nullity, we conclude that the rank of D on $\mathcal{P}_{N+r}$ equals N, and hence the entire $\mathcal{P}_N$ is the range. Since N is arbitrary, every polynomial is in the range of D.

9. Prove that every polynomial $p(\lambda)$ of degree 2015 with rational coefficients has a "root" $\lambda = A$ (i.e. $p(A)=0$) among square matrices A of size 2015 with rational entries.

`Solution.` For $p = a_0\lambda^{2015} + a_1\lambda^{2014} + \cdots + a_{2015}$, take

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \\ 0 & & \cdots & 0 & 1 \\ -\frac{a_{2015}}{a_0} & & \cdots & & -\frac{a_1}{a_0} \end{bmatrix}.$$

Then $p(\lambda)/a_0$ is the characteristic polynomial of A, and $p(A)=0$ by the Cayley–Hamilton identity.

10. Let U be a linear operator on $\mathbb{C}^n$ such that $U^{2015} = I$. Prove that U is diagonalizable, i.e. $\mathbb{C}^n$ has a basis consisting of eigenvectors of U.

`Solution.` Start with any Hermitian inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^n$ and take the average of its transformations by powers of U:

$$(x, y) := \frac{1}{2015} \sum_{k=0}^{2014} \langle U^k x, U^k y \rangle.$$

Then $(Ux, Uy) = (x, y)$ for any $x, y \in \mathbb{C}^n$, i.e. U is unitary with respect to $(\cdot, \cdot)$. By the Spectral Theorem, U has an (orthonormal) basis of eigenvectors.

11. Prove that if all eigenspaces of an operator A on $\mathbb{C}^n$ are 1-dimensional, then the space of all operators B commuting with A has dimension n.

`Solution.` An operator B commuting with A preserves the root spaces of A. When all eigenspaces of A are 1-dimensional, different Jordan cells of A have different eigenvalues. Therefore B is block-diagonal in the Jordan basis of A, and it suffices to check that matrices commuting with a Jordan cell of size r (the eigenvalue can be taken equal to 0, since scalar matrices commute with any ones) form an r-dimensional space. A straightforward computation shows that C commutes with the size-r regular nilpotent Jordan cell if and only if C has the form

$$\begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_r \\ 0 & c_1 & c_2 & c_3 & \dots \\ 0 & 0 & c_1 & c_2 & \dots \\ & & \dots & & \\ 0 & \dots & 0 & c_1 & c_2 \\ 0 & 0 & \dots & 0 & c_1 \end{bmatrix}.$$

12. For an $n \times n$-matrix A of rank $n-1$, find the rank of the cofactor matrix $adj(A)$.

`Solution.` When $\operatorname{rk}(A) = n - 1$, the cofactor matrix is non-zero (since A has a non-zero $(n-1) \times (n-1)$-minor), but $\det A = 0$. Then by the Cofactor Theorem, $adj(A)A = (\det A)I = 0$, and the null-space of $adj(A)$ must contain the $(n-1)$-dimensional range of A. Thus $1 \leq \operatorname{rk}(adj(A)) \leq n - (n-1) = 1$, i.e. the rank of $adj(A)$ equals 1.