Math 53. Multivariable Calculus.
Final Exam. 12.15.15. Solutions

1. Change the order of integration:

$$\int_0^1\int_0^{\sqrt{z}}\int_0^{\sqrt{y}} f(x,y,z)\,dxdydz = \int_?^?\int_?^?\int_?^? f(x,y,z)\,dydzdx\,.$$

`Solution.` The region is $0\le x^4\le y^2\le z\le 1$. Respectively the integral is equal to

$$\int_0^1\int_{x^4}^1\int_{x^2}^{\sqrt{z}} f(x,y,z)\,dydzdx.$$

2. In what proportion is the surface of the sphere of radius r divided by a plane passing at the distance $d<r$ from the center? Justify your answer.

`Solution.` There are several ways to compute the area of the part of the sphere $x^2+y^2+z^2=r^2$ above the plane $z=d$.

A. The area of the graph of the function $z(x,y)=\sqrt{r^2-x^2-y^2}$ over the domain $x^2+y^2\le r_0^2:=r^2-d^2$ is found by integrating

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+\frac{x^2}{r^2-x^2-y^2}+\frac{y^2}{r^2-x^2-y^2}}=\frac{r}{\sqrt{r^2-x^2-y^2}}.$$

In polar coordinates $\delta=\sqrt{x^2+y^2},\theta=\arctan\frac{y}{x}$, we find:

$$r\int_0^{2\pi}\int_0^{r_0}\frac{\delta\,d\delta\,d\theta}{\sqrt{r^2-\delta^2}}=-\frac{2\pi r}{2}\int_0^{r_0}\frac{d(r^2-\delta^2)}{\sqrt{r^2-\delta^2}}=-2\pi r\sqrt{r^2-\delta^2}\,\Big|_{\delta=0}^{\delta=r_0}$$

$=2\pi r(r-d)$, since $r_0^2=r^2-d^2$, and hence $\sqrt{r^2-r_0^2}=d$.

B. Alternatively, in spherical coordinates the surface are of this piece of the sphere is given by the formula

$$r^2\int_0^{2\pi}\int_0^{\arccos d/r}\sin\phi\,d\phi d\theta=2\pi r^2\left(1-\frac{d}{r}\right).$$

C. Perhaps, the most illuminating way is based on representing the sphere as the surface of revolution about the z-axis with the cylindrical equation $\delta(z)=\sqrt{r^2-z^2}$, whose surface area is given by the integral

$$2\pi\int_d^r\sqrt{1+\delta_z^2}\,\delta(z)\,dz=2\pi r\int_d^r dz=2\pi r(r-d).$$

The rest of the sphere has the area $2\pi r(r+d)$. Thus, the required ratio is $(r-d)/(r+d)$.

3. To a table's surface with coordinates (x, y), a square piece of rubber $0 \le u, v \le 1$ is glued according to the rule

$$x(u,v) = 3u + v^2, \quad y(u,v) = v + (3u + v^2)^2.$$

Find the area of the region covered by this piece of rubber.

`Solution.` The Jacobian

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3 & 2v \\ 2(3u+v^2)3 & 1 + 2(3u+v^2)2v \end{vmatrix} = 3,$$

and hence the area of the *image* of the square is 3 times the area of the square, i.e. 3.

4. Find the maximum and minimum values of the function $2x^2 + 4xy - y^2$ on the unit circle $x^2 + y^2 = 1$.

`Solution.` By the Lagrange multiplier method, we get the system of two linear equations

$$4x + 4y = 2\lambda x, \ 4x - 2y = 2\lambda y,$$

which has solutions under the constraint $x^2 + y^2 = 1$ only when

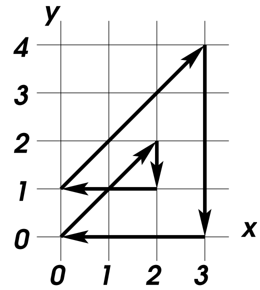
$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0,$$

i.e. $\lambda = 3$ or $\lambda = -2$. The solutions have $x/y = 2$ and $y/x = -2$ respectively. The respective critical points are: $(x, y) = \pm(2/\sqrt{5}, 1/\sqrt{5})$ and $(x, y) = \pm(1/\sqrt{5}, -2/\sqrt{5})$. The corresponding critical values are 3 and -2. Therefore the maximum value is 3, and the minimum -2.

5. Does there exist a function $f(x, y)$ satisfying $f_{xx}(0, 0) = 8$, $f_{xy}(0, 0) = 4$, $f_{yy}(0, 0) = 2$ which at $(x, y) = (0, 0)$ has: (a) a local minimum? (b) a local maximum? (c) a critical point which is neither local minimum nor local maximum? In each case (a),(b),(c), if the answer is "no" explain "why", if "yes", give an example of such a function and explain why the required property holds.

`Solution.` (a) $g(x, y) := 4x^2 + 4xy + y^2 = (2x + y)^2$ has the required values of g_{xx}, g_{xy}, g_{yy}, and a local minimum at the origin. (b) Since $f_{xx}(0, 0) = 4 > 0$, the function $f(x, 0)$ has an isolated local minimum at $x = 0$, and hence $f(x, y)$ cannot have a local maximum at the origin. (c) $f(x, y) := (2x + y)^2 - (x^4 + y^4)$ on the line $2x + y = 0$ is negative unless $(x, y) = (0, 0)$, and hence f has neither minimum nor maximum at the origin.

6. A Microsoft Mouse crawls on the grid mousepad from the origin $(0,0)$ straight to the point $(2,2)$, then to $(2,1)$, then to $(0,1)$, then to $(3,4)$, then to $(3,0)$, and then back to $(0,0)$. Compute the total work that the force field $V = (xy-1)y\,\vec{\mathbf{i}} + x(xy+1)\,\vec{\mathbf{j}}$ performs on the Mouse during this voyage.



`Solution.` Taking $Q = (xy-1)y$ and $P = x(xy+1)$, we find $Q_x - P_y = 2$. By Green's Theorem, the work is *twice* the signed area enclosed by the path. Since during the voyage, the region stays on the *right* of the Mouse, the sign is negative. Since the small triangle is enclosed twice, the total area is 7.5. Thus the work is -15.

7. Show that the curl of a vector field has zero flux across any sphere contained in the domain of the vector field, and give an example of a divergence-free vector field which has non-zero flux across this sphere.

`Solution.` By Stokes' Theorem, for a sphere S

$$\iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{dS} = \int_{\partial S} \mathbf{V} \cdot \mathbf{dl} = 0$$

since $\partial S = 0$. If $\mathbf{r}_0$ is a point inside the sphere (e.g. the center), the flux of the divergence-free vector field $\mathbf{r} - r_0/|\mathbf{r} - r_0|^3$ across the sphere will be equal to $]pm4\pi$ (depending on the orientation), i.e. non-zero.

8. Compute the flux of vector field

$$V = \frac{x}{x^2+y^2}\,\vec{\mathbf{i}} + \frac{y}{x^2+y^2}\,\vec{\mathbf{j}} + 0\,\vec{\mathbf{k}}$$

across the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

oriented by the exterior normal.

`Solution.` We have: $V = \vec{r}/|\vec{r}|^2$, where $\vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$. Therefore

$$\nabla \cdot V = \frac{\nabla \cdot \vec{r}}{|\vec{r}|^2} + \vec{r} \cdot \nabla \frac{1}{|\vec{r}|^2} = \frac{2}{|\vec{r}|^2} - \frac{2\vec{r} \cdot \vec{r}}{|\vec{r}|^4} = 0.$$

Yet, V is not defined on the z-axis. An attempt to apply Gauss' Theorem to the region enclosed by the ellipsoid is illegal (and gives incorrect result 0). Instead, take the region R between the surface of the ellipsoid and a cylinder of large radius $r > a, b$ around the z-axis, whose top and bottom touch the ellipsoid at its points $\pm(0, 0, c)$ on the z-axis (see Figure). By Gauss' Theorem,

$$\iint_{\partial R} V \cdot \mathbf{d}S = \iiint_R \nabla \cdot V \, dx \, dy \, dz = 0.$$

Therefore the flux of V across the ellipsoid is equal to the flux across the cylinder. Since V has zero z-component, its flux across the top and the bottom of the cylinder vanishes. The flux across the lateral surface of the cylinder, with normal vector is $\vec{n} = \vec{r}/|\vec{r}|$, is easily computed as

$$\int_{-c}^{c} \oint_0^{2\pi} \frac{\vec{r}}{r^2} \cdot \frac{\vec{r}}{r} r d\theta \, dz = 2\pi \int_{-c}^{c} dz = 4\pi c.$$