Math 214: Differential Manifolds. Take Home Final

Rules. *Pick 3 out of 6 problems (but not more) and solve them. Your solutions are due at 5 p.m. on Thursday, Dec. 13 (put them into my mailbox, or under my door). In your work on the problems, you may not consult with other people, the internet, or library. You may use however the course textbook and your notes.*

1. Prove that the Grassmann manifold $Gr^+_{2,4}(\mathbf{R})$ of oriented planes in $\mathbf{R}^4$ is diffeomorphic to $S^2 \times S^2$.

2. Find out which of the Grassmann manifolds $Gr_{k,n}(\mathbf{R})$ are orientable.

3. Prove that a compact Lie group carries a volume form invariant to both right and left translations, and give an example of a non-compact Lie group that doesn't.

4. Consider $\omega := x_1 \wedge y_1 + \cdots + x_n \wedge y_n$ as an anti-symmetric bilinear form in $\mathbf{R}^{2n}$. In the $2n-1$-dimensional projective space $P(\mathbf{R}^{2n})$, define the following $2n-2$- dimensional distribution $\mathcal{D}$: given a point $l \in P(\mathbf{R}^{2n})$, consider the ω-orthogonal complement to $l \subset \mathbf{R}^{2n}$ (which is a $2n-1$-dimensional subspace L containing l) and take on the role of $\mathcal{D}(l)$ the tangent space to the hyperplane $P(L) \subset P(\mathbf{R}^{2n})$ at the point l.

Find the maximal dimension of integral submanifolds of this distribution.

5. Prove that every two non-vanishing closed differential $n-1$-forms in $\mathbf{R}^n$ are locally diffeomorphic.

6. Prove that a smooth function in n variables in a neighborhood of a critical point where the quadratic differential has the rank r is locally diffeomorphic to a function of the form

$$x_1^2 + \cdots + x_r^2 + f(y_1, \ldots, y_{n-r}),$$

where f has the zero quadratic differential at the critical point $y = 0$.