EXERCISES IN SYMPLECTIC GEOMETRY [1]

1. Prove the Linear "Relative Darboux Theorem" (page 6).
2. Verify that the graph of a linear map $A : X \to X^*$ is Lagrangian in $V = X^* \oplus X$ if and only if A is symmetric: $A^* = A$.
3. Write explicitly the Plücker relation $\phi \wedge \phi = 0$ for an exterior 2-form $\phi = \sum_{1 \leq i<j \leq 4} \phi_{ij} x_i \wedge x_j$ in $\mathbb{R}^4$ and show that it is a non-degenerate quadratic equation of signature $(+++---)$. Derive that the grassmannian $Gr_{2.4}(\mathbb{R})$ is diffeomorphic to the quotient of $S^2 \times S^2$ by the simultaneous antipodal map $(x, y) \mapsto (-x, -y)$, and give the analogous description of the Lagrange grassmannian Λ_2.
4. Verify the formula (on page 8) for the Poisson bracket of quadratic Hamiltonians in Darboux coordinates.
5. Show that the adjoint action of the symplectic group on its Lie algebra coincides with the its action by the changes of variables in the quadratic hamiltonians. Derive from this that the center of $Sp(2n, \mathbb{R})$ consists of two elements $\pm E_{2n}$.
6. Prove that the one-parametric subgroup in $Sp(2n, \mathbb{R})$ generated by a positive definite quadratic Hamiltonian is pre-compact.
7. For $\omega = \sum \lambda_l p_k \wedge q_k$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$, show that the ω-area of the unit disk on any oriented plane L (with respect to the Euclidean norm $\sum(p_k^2 + q_k^2)$ in the ambient space) does not exceed $\pi\lambda_1$. (*Hint:* First show this this for $\lambda_1 = ... = \lambda_n = 1$ using the Cauchy–Schwarz inequality for the Hermitian form $\langle z, w \rangle = \sum z_k \bar{w}_k$.)
8. Show that an infinitesimal symplectic transformation $H \in sp(V)$ is similar to $-H$ (and therefore its eigenvalue spectrum and the whole Jordan block structure is invariant under the transformation $\lambda \mapsto -\lambda$).
9. Show that the symmetric operator S in the polar factorization $A = SU$ of a linear symplectic transformation A, in a suitable decomposition $\oplus_{k=1}^n \mathbb{R}^2$ of the symplectic space into the direct sum of n orthogonal (with respect to both the symplectic and Euclidean structures) symplectic planes, is described as the superposition of symplectic hyperbolic rotations in these planes (or, equivalently, as the time-1 map in the flow of the Hamiltonain $\sum_{k=1}^n \lambda_k p_k q_k$ with suitable real λ_k).
10. Linearize the pendulum equation near the upper (unstable) equilibrium, sketch the phase portrait, and describe the phase flow in terms of hamiltonian hyperbolic rotations.
11. Describe (and sketch) the partition of the 3-dimensional space of quadratic forms on the symplectic plane $(\mathbb{R}^2, p \wedge q)$ into the orbits of the adjoint

[1] Math 242, Fall'21, A. Givental, references to "Symplectic geometry" by Arnold–Givental in vol. 4 of Springer's EMS series.

action of $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$. Indicate in your sketch the region of sign-definite quadratic hamiltonians.

12. Show that the $\mathbb{H}$-valued positive-definite sesquilinear form in $\mathbb{H}^n$ (with *right* multiplication by quaternionic scalars) which is given by the formula $\langle x, y\rangle := \sum_{m=1}^n x_m^* y_m$ has the structure $Z(x,y) + jW(x,y)$ where Z and W are respectively Hermitian and complex symplectic forms on $\mathbb{H}^n = \mathbb{C}^{2n}$ with respect to the complex structure defined by i. Express Z via W (and j) and *vice versa* W via Z (and j). Derive that $GL_n(\mathbb{H}) \cap U_{2n}$ (where $U_{2n} = Aut_{\mathbb{C}}(Z)$) coincides with $GL_n(\mathbb{H}) \cap Sp(2n,\mathbb{C})$ (where $Sp(2n,\mathbb{C}) = Aut_{\mathbb{C}}(W)$ and therefore coincides with the compact groups Sp_n of quaternionic-linear automorphismsm of the $\mathbb{H}$-valued sesquilinear form $\langle\cdot,\cdot\rangle$. Show that this is a maximal compact subgroup in $Sp(2n,\mathbb{C})$, and find its dimension and rank (i.e. the dimension of a maximal torus T), its normalizer $N(T)$, and the corresponding Weyl group $W := N(T)/T$.

13. Write down miniversal deformations of each of the two quadratic hamiltonians $(p_1^2 + q_1^2) \pm (p_2^2 + q_2^2)$, and describe the subset in the parameter space consisting of *stable* hamiltonians, (i.e. consisting of *diagonalizable* infinitesimal symplectic transformations with purely imaginary spectrum).

14. Which of the following three groups are isomorphic, and which are not: (a) $Sp(2,\mathbb{R})$, (b) $SU(1,1)$ (the group of unimodular complex 2×2-matrices preserving the indefinite Hermitian form $|z_1|^2 - |z_2|^2$), (c) the group of *real* Möbius transformations $w = (az+b)/(cz+d)$ of the Riemann sphere?

15. Prove that the space of complex structures $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $J^2 = -E_{2n}$ in symplectic $\mathbb{R}^{2n}$ such that the symplectic form ω is the imaginary part of a (positive definite) Hermitian form (and hence such that $\omega(\cdot, J\cdot)$ is its real part) is contractible. (*Hint:* Identify the space with Siegel's "upper half-plane" $Sp(2n,\mathbb{R})/U_n$.)

16. Show that the structure of a real symplectic vector bundle can be upgraded to the structure of an Hermitian vector bundle (with the imaginary part of the Hermitian form being the given symplectic one). Derive that classification of real symplectic $2n$-dimensional vector bundles over a given base coincides with the classification of complex n-dimensional vector bundles over that base.

17. Prove "The Extension Theorem II" from page 26: A smooth field Ω of nondegenerate 2-forms on $T_N M$ whose restriction to TN define a closed differential 2-form ω on N can be extended to a symplectic structure to a neighborhood of N in M. (*Hint:* Use tubular neighborhood projection $\pi : M \to N$, and look for a differential 1-form α on M vanishing at the points of N and such that $d\alpha|_{T_N M} = \Omega - \pi^*\omega|_{T_N M}$, by applying a partition of unity on N to the RHS, and constructing α locally.)

18. Prove Moser's volume theorem: On a compact connected oriented manifold (without boundary) two volume forms with the same total volume can be transformed into each other by a diffeomorphism (and obviously *vice versa*.

19. The same — for two symplectic structures which can be connected by a continuous family of symplectic structures within the same De Rham cohomology class.

20. Use Moser's homotopy method to prove the Morse lemma: A germ of function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ at a non-degenerate critical point can be transformed by a germ of diffeomorphism $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ into the quadratic form of its 2nd differential $d_0^2 f$.

21. Establish a one-to-one correspondence between equivalence classes of complex vector bundles over a given manifold L and local symplectormorphism classes of isotropic embeddings of L into symplectic manifolds (more precisely, understanding the equivalence of two isotropic embeddings as a symplectomorphism between sufficiently small neighborhoods of L in the ambient symplectic manifolds which is identical on L).

22. Show that the *conormal* bundle N_X of a submanifold $X \subset M$ (by definition N_X is the subset in T^*M consisting of all covectors applied at a point of X and vanishing on tangent vectors to X at that point) is a Lagrangian submanifold in T^*M with respect to the canonical sumplectic structure. Compute the conormal bundle for the semicubical parabola $y^2 = x^3$ on the plane, first at non-nonsingular points, but then take the closure in $T^*\mathbb{R}^2$. Try to describe the resuting (singular) Lagrangian surface as an abstract algebraic variety (e.g. by characterizing somehow the ring of polynomial functions on it, or in any other way).

23. Compute adjoint and coadjoint orbits of the group of affine transformations on the line: $x \mapsto ax + b$.

24. Show that coadjoint orbits of the Lie algebra $\mathbb{R}^3$ with respect to the cross-product are concentric spheres $F = const$, where $F = (x^2 + y^2 + z^2)/2$ in Euclidean coordinates, and identify the symplectic (area) form on the orbits with the *Leray form* $(dx \wedge dy \wedge dz)/dF$ (an expression defining correctly a differential 2-form on the level sets of F at non-singular points).

25. When two vector fields u, v on a symplectic manifolds correspond to two closed 1-forms, the (exact) 1-form corresponding to their Lie bracket $[u, v]$ is $di_u i_v \omega$. Generalize this formula to the Lie bracket of two vector fields u and v corresponding to arbitrary (non-necessarily closed) differential 1-forms α and β.

26. Let v be a vector field on X. Its flow lifted naturally to T^*X preserves the canonical symplectic structure. Show that the Hamilton function of this flow is given by the rule: Its value at a point $p \in T_q^*X$ is equal to $p(v(q))$.

27. Prove that functions on T^*X, linear on the cotangent fibers, form a Lie subalgebra with respect to the Poisson bracket, which is canonically isomorphic to the Lie algebra of vector fields on the base.

28. Prove that a submanifold in a symplectic manifold is coisotropic if and only if the ideal of smooth functions vanishing on it is a Poisson subalgebra (i.e. closed with respect to the Poisson bracket).
29. Let $S \supset P$ be the germ of a symplectic leaf in a Poisson manifold P, and I the ideal of S in the algbera $C^\infty(P)$ of germs of smooth functions on P. Show that I (as well as every power I^k) is a Lie ideal with respect to the Poisson bracket, and that the induced Poisson bracket on I/I^2 endows the fibers of the (conormal to S) vector bundle $(T_S P/TS)^*$ with the structure of a Lie algebra (more precisely, the space of sections becomes a Lie algebra over the ring $C^\infty(S)$). Show that all Lie algebras in this family (over the base S) are isomorphic to each other (and to the linear approximation of the transverse Poisson structure of S).
30. Take ξ to be the Jordan normal form of subregular nilpotents in $\mathfrak{g}^* = gl_{n+1}^* \equiv gl_{n+1}$, and compute coadjoint orbits of the Lie algebra $\mathfrak{g}_\xi$ (the annihilator of ξ). Show (relying on the computations done in the lecture) that the foliation of $\mathfrak{g}_\xi^*$ into coadjoint orbits is not isomorphic to the transverse Poisson structure of ξ in $\mathfrak{g}^*$.
31. In the parameter plane (with coordinates c_2, c_3) of the family of surfaces $xy + z^3 + c_2 z + c_3 = 0$ (foliating the transversal Poisson manifold of the subregular nilpotent $\xi \in sl_3^*(\mathbb{C})$ into the closures of symplectic leaves), compute and sketch the discriminant, i.e. the image of the leaves of dimension 0.
32. Check that the De Rham diferential $d : \Omega^0(S^1) \to \Omega^1(S^1)$ is anti-symmetric with respect to the pairing $(f, \alpha) := \frac{1}{2\pi} \oint f\alpha$ (between functions f and 1-forms α), and hence defines a (translationally invariant) Poisson structure on Ω^1. Show that $c := (1, \alpha)$ is Casimir, and that $c = consts$ are the symplectic leaves. More specifically (assuming that the functions and 1-forms are complex-calued), show that the Poisson brackets between the Fourier coordinates $q_k := (e^{-ikx}, \alpha)$, $p_k := (e^{ikx}, \alpha)$, $k = 1, 2, \ldots,$ are all zeros except $\{p_k, q_k\} = ik = -\{q_k, p_k\}$.
33. Prove that the fixed point locus of an anti-symplectic involution of a symplectic manifold is a Lagrangian submanifold (if non-empty).
34. Let a smooth map $\pi : (M^{2n}, \omega) \to X^n$ of a $2n$-dimensional compact symplectic manifold to an n-dimensional manifold be such that $\pi^* C^\infty(X)$ is a commutative subalgebra in $C^\infty(M)$ with respect to the Poisson bracket, and let $X_0 \subset X$ be a (non-empty) connected component of the regular value set of π. Prove that $\pi^{-1}(X_0) \xrightarrow{\pi} X_0$ is a Lagrangian fibration, and describe possible fibers of it.
35. For the natural mechanical system with the Lagrangian $L = T - U$, show directly that $T + U$ is constant along the solutions of the Euler-Lagrange equation.

36. Show that trajectories of a free particle on a Riemannian manifold are geodesics parameterized to a constant speed relative to the arc length.
37. Let ω be an element of the Lie algebra of SO_n (i.e. an anti-symmetric matrix), and let ρ be the mass density of a rigid body in the Euclidean space $\mathbb{R}^n$. Express the kinetic energy $T(\omega)$ of the rotation $t \mapsto \exp \omega t$ as a quadratic form on the Lie algebra. Show that for $n = 3$, every left-invariant Riemannian metric on the group is thus obtained, but for $n > 3$ this is no longer the case.
38. Let $\vec{B}$ be a (divergence-free) magnetic vector field in $\mathbb{R}^3$, and $\vec{B} \times \dot{q}$ be the magnetic force acting on the unit charge moving with velocity $\dot{q}$. Let $B := i_{\vec{B}} dx \wedge dy \wedge dz$ be the closed differential 2-form corresponding to $\vec{B}$, and A-the (locally defined) differential 1-form such that $dA = B$ (the "vector-potential" of the magnetic field). Show that including the term $\int A$ into the Lagrangian results in adding to the Euler-Lagrange equation the above magnetic force (in the form $i_{\dot{q}} B$).
39. Find trajectories of a mass-1 particle on the xy-plane in the "magnetic field" $dx \wedge dy$.
40. Apply Noether's theorem to a free particle on a Lie group G (i.e. taking the Lagrangian to be the kinetic energy invariant under left translations on G), and derive the corresponding conservation laws.
41. Write down the Euler-Poisson equation and the corresponding hamiltonian for the functional $\int [m(d^n x/dt^n)^2/2 - U(x)]\, dt$.
42. Show that the quadratic hamiltonian $H = \sum_{i=1}^n \omega_i (p_i^2 + q_i^2)$ is completely integrable, and describe explicitly the corresponding action-angle variables.
43. Show that the kinetic energy of a rigid body in $\mathbb{R}^n$ (which is a quadratic form on so_n) in a suitable coordinate system in $\mathbb{R}^n$ can be described as $T(\omega) = -\operatorname{tr}(\omega D \omega)$, where D is a diagonal matrix with non-negative diagonal entries $d_1, \dots, d_n$, and ω is the anti-symmetric matrix representing in this coordinate system an infinitesimal rotation of the body.
44. Show how to "integrate in quadratures" a hamiltonian system on the symplectic plane.
45. Suppose that (some connected components of) the level curves $H(p, q) = h$ of a given Hamilton function on the symplectic plane form a family of ovals, which therefore represent periodic solutions. Let $T(h)$ be the period, and $S(h)$ be the symplectic area enclosed by the oval at the level h. Show that $T = dS/dh$.
46. Show that the conformal mapping $z \mapsto w = (z + 1/z)/2$ transforms concentric circles and perpendicular to them radial rays on the z-plane into ellipses and (branches) of hyperbolas on the w-plane with the common foci ± 1.
47. Show that the λ-family $x^2/(a-\lambda) + y^2/(b-\lambda) + z^2/(c-\lambda) = 1$ ($a > b > c$) of "confocal" surfaces contains one ellipsoid, one one-sheeted hyperboloid,

and one two-sheeted hyperboloid passing through every given typical point in space.

48. Check that the Lax equation $\dot{L} = [L, A]$, where $L := -d^2/dx^2 + u$ and $A := d^3/dx^3 - 3\,(u\; d/dx + d/dx\; u)$, is equivalent to the KdV equation $\dot{u} = 6uu_x - u_{xxx}$.

49. Show that the KdV equation is bi-hamiltonian: with the Hamilton functional $H_0[u] = \int \frac{u^2}{2} dx$ and Poisson operator $W := 2\left(u\frac{d}{dx} + \frac{d}{dx}u\right) - \frac{d^3}{dx^3}$, and with the Hamilton functional $H_1[u] = \int \left(\frac{u_x^2}{2} + u^3\right) dx$ and the Poisson operator $V := \frac{d}{dx}$. Check directly that H_0 is a conservation law of the KdV flow. For the W-hamiltonian flow with the Hamilton functional H_1, find the V-hamiltonian H_2. (According to the Lenard-Magri scheme, it is another conservation law of the KdV flow, Poisson-commuting with H_0 and H_1 with respect to both Poisson brackets V and W.)

50. Identify the complexified Lie algebra $Vect(S^1)$ of (polynomial in trigonometric sense) vector fields on the circle with the Lie algebra described by the relations $[L_m, L_n] = (m-n)L_{m+n}$ in a certain basis L_m, $m = 0, \pm 1, \pm 2, \ldots$. Show that $\omega(L_m, L_n) := m^3 \delta_{m+n,0}$ defines a 2-cocycle on $Vect(S^1)$. Check that $Span(L_{-1}, L_0, L_1)$ is a Lie subalgebra, on which this cocycle is a coboundary. Can you explain what group of diffeomorphisms of S^1 corresponds to this 3-dimensional Lie subalgebra?

51. Let (M, ω) be a compact connected symplectic manifold, and x_0 a point in M. Given two hamiltonian vector fields v, w on M, put $C(v, w) := \omega(v(x_0), w(x_0))$. Show C is a 2-cocycle the Lie algebra of hamiltonian vector fields on M. What central extension is defined by this cocycle. Is this cocycle a coboundary?

52. Check the claim of Example 2 on p. 62: The moment map $T^*G \to \mathfrak{g}^*$ of the action of G on T^*G defined by the left translations on G coincides with the right translations of covectors to $\mathfrak{g}^* = T_e^*G$.

53. Let the Lie algebra $\mathfrak{g}$ of a Lie group G be equipped with a non-degenerate symetric bilinear form $\mathrm{tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. On the affine space of connections $\nabla = d + A\wedge : \Omega^0(\Sigma, \mathfrak{g}) \to \Omega^1(\Sigma, \mathfrak{g})$ on the trivial G-bundle over a compact oriented 2-dimensional manifold Σ, define a translation-invariant symplectic structure $\omega(\alpha, \beta) := \int_\Sigma \mathrm{tr}(\alpha \wedge \beta)$, where $\alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g})$ are $\mathfrak{g}$-valued differential 1-forms on Σ (so that $\alpha \wedge \beta \in \Omega^2(\Sigma, \mathfrak{g} \otimes \mathfrak{g})$ and $\mathrm{tr} : \Omega^2(\Sigma, \mathfrak{g} \otimes \mathfrak{g}) \to \Omega^2(\Sigma, \mathbb{R})$). Show that the moment map of the action $\nabla \mapsto g^{-1} \nabla g$ of the gauge group $C^\infty(\Sigma, G)$ on the space of connections is Poisson, and that its moment map associates to a connection its curvature: $\nabla \mapsto \nabla^2 = dA + A \wedge A \in \Omega^2(\Sigma, \mathfrak{g})$.

54. Let $H \subset G$ be a connected Lie subgroup in a connected Lie group. Show that the moment map of the H-action on coadjoint G-orbits is given by the projection $\mathfrak{g}^* \to \mathfrak{h}^*$ dual to the embedding $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebras. Take

H to be the subgroup of diagonal matrices in $G = SU_3$, and compute (and sketch!) the images of the coajoint orbits under the moment map.

55. The action of the group T^{n+1} of unitary diagonal $n+1$-matrices on $\mathbb{C}^{n+1}$ descends to $\mathbb{C}P^n = proj(\mathbb{C}^{n+1})$. Show that the latter action is Poisson (with respect to the Fubini-Study symplectic structure on $\mathbb{C}P^n$ considered as a real manifold) and find the image of its moment map.

56. Given a Poisson action of T^k on a compact connected symplectic manifold M, consider its moment map $f = (f_1, \ldots, f_k) : M \to \mathbb{R}^k = Lie^*(T^k)$ and the (finite) set of common critical values $c_\alpha \in \mathbb{R}^k$ of the hamiltonians $f_1, \ldots, f_k$. Prove that critical values of the map f lie in the union of finitely many affine hyperplanes spanned by (some of) the points c_α.

57. Consider the Poisson action of T^{n+1} on $\mathbb{C}P^n = proj(\mathbb{C}^{n+1})$ (as in Exercise 55), and let H_i, $i = 0, \ldots, n$ be the hamiltonians generating the actions. Prove that

$$\int_{\mathbb{C}P^n} e^{\sum u_i H_i} \frac{\omega^n}{n!} = \frac{1}{2\pi\sqrt{-1}} \oint \frac{e^p dp}{\prod_i (p - u_i)},$$

where the residue integral is taken over a contour enclosing all the $n+1$ poles $p = u_i$.

58. Prove that a contact 3-dimensional manifold is orientable.

59. The *front* of a Legendrian submanifold $L \subset PT^*X$ is defined as the image $\pi(L) \subset X$ of its projection to the base. Prove that the front determines the Legendrian submanifold — at least in the typical case when the front is a hypersurface (but also when the front has higher codimension — in the narrower sense that L is an open submanifold in the *maximal* Legendrian submanifold with the front $\pi(L)$).

60. Prove that a neighborhood of a closed Legendrian submanifold $L \subset (N, \Pi)$ is contactomorphic to a neighborhood of the zero section in the space of 1-jets of sections of the line bundle over L whose fiber at $x \in L$ equals $T_x N / \Pi(x)$.

61. For a Lagrangian circle $p^2 + q^2 = 1$ on the symplectic (p, q)-plane, find the corresponding Legendrian curve in the contactization $du = pdq$ and sketch the front in the (u, q)-plane (i.e. the image of the Legendrian projection $(u, p, q) \mapsto (u, q)$.

62. Using symplectizations, prove that contact 1-forms in $\mathbb{R}^{2n-1}$ are locally diffeomorphic.

63. Prove the contact analogue of Moser's stability theorem: On a closed manifold, any two contact structures within the same connected component in the space of all contact structures are contactomorphic.

64. By (polynomial) differential operators in $\mathbb{R}^n$, we mean non-commutative polynomials $D(\partial_q, q)$. Check that the associative algebra $\mathcal{D}$ of such operators is filtered by operators' order (i.e. if $\mathcal{D}_n$ denotes the space of such operators of order $\leq n$, then $\mathcal{D}_m \mathcal{D}_n \subset D_{m+n}$). Identify the associated graded algebra

$Gr(\mathcal{D}) := \oplus_{n=0}^{\infty}\mathcal{D}_n/\mathcal{D}_{n-1}$ with the (commutative!) algebra of polynomial functions on $T^*\mathbb{R}^n$. In particular, show that for $D_m \in \mathcal{D}_m$ and $D_n \in \mathcal{D}_n$, the commutator $[D_m, D_n]$ lies in $\mathcal{D}_{m+n-1}$. Prove that the commutator operation induces on $Gr\mathcal{D}$ the operation of Poisson bracket of polynomial functions on $T^*\mathbb{R}^n$.

`Remark.` The meaning of this exercise is that the algebra of quantum observables (strictly speaking, they should have the form $D(-i\hbar\partial_q, q, \hbar)$ where $\hbar = h/2\pi$, and be filtered by degrees in $\hbar$) in the classical limit $\hbar = 0$ turns into the Poisson algebra of classical observables. The same is true for smooth differential operators on a manifold X if by classical observables one means functions on T^*X polynomial along the fibers.

65. Starting from the prequantization of the symplectic plane $\mathbb{R}^2, dp \wedge dq$ given by the contact 1-form $\theta = du + pdq$, find the quantizations $\hat{p}$ and $\hat{q}$ of the generators of the Heisenberg algebra (by definition, it consists of Hamiltonians of degree ≤ 1) in the "impulse" representation, i.e. in the space of sections of the Hermitian $\mathbb{C}$-bundle over the symplectic plane which are covariantly constant along the Lagrangian lines $p = const$.

66. Use the method of characteristics to find the solution to Burgers' equation $u_t = uu_x$ satisfying the initial condition $u(t,x)_{t=0} = v(x)$.