EXPLICIT RECONSTRUCTION IN QUANTUM COHOMOLOGY AND K-THEORY

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Abstract. Cohomological genus-0 Gromov-Witten invariants of a given target space can be encoded by the "descendant potential," a generating function defined on the space of power series in one variable with coefficients in the cohomology space of the target. Replacing the coefficient space with the subspace multiplicatively generated by degree-2 classes, we explicitly reconstruct the graph of the differential of the restricted generating function from one point on it. Using the Quantum Hirzebruch–Riemann–Roch Theorem from our joint work [8] with Valentin Tonita, we derive a similar reconstruction formula in genus-0 quantum K-theory. The results amplify the role of the divisor equations.

1. Formulations

Let X be a compact Kähler (or, more generally, symplectic) manifold. Its genus-0 *descendant potential* is defined by

$$\mathcal{F}(t) := \sum_{d\in\mathcal{M}}\sum_{n=0}^{\infty}\frac{Q^d}{n!}\langle t(\psi),\dots,t(\psi)\rangle_{0,n,d},$$

where $\mathcal{M}\subset H_2(X,\mathbb{Z})$ is the Mori cone of X, Q^d stands for the element corresponding to d in the semigroup ring of $\mathcal{M}$, $t:=\sum_{k\geq 0}t_kz^k$ is a power series with coefficients t_k which are cohomology classes of X, and the correlator stands for the integral over the virtual fundamental class $[X_{0,n,d}]$ of the moduli space of degree-d stable maps to X of rational curves with n marked points:

$$\langle\phi_1\psi^{k_1},\dots,\phi_n\psi^{k_n}\rangle_{0,n,d} := \int_{[X_{0,n,d}]}\mathrm{ev}_1^*(\phi_1)\psi_1^{k_1}\cdots\mathrm{ev}_n^*(\phi_n)\psi_n^{k_n}.$$

Here ev_i^* is the pull-back of cohomology classes from X to $X_{0,n,d}$ by evaluiation map at the i-th marked point, and ψ_i is the 1st Chern class of the line bundle over $X_{0,n,d}$ formed by cotangent lines to the curves at the i-th marked point.

This material is based upon work supported by the National Science Foundation under Grant DMS-1007164.

Following [6], we embed the graph of the differential of $\mathcal{F}$ into the *symplectic loop space* $\mathcal{H}$. By definition, it consists of formal Q-series with coefficients whose coefficients are Laurent series in one indeterminate z with vector coefficients from $H^*(X,\mathbb{Q})$.

The "loop space" $\mathcal{H}$ (which is actually a $\mathbb{Z}_2$-graded module over the Novikov ring $\mathbb{Q}[[Q]]$) is equipped with the $\mathbb{Q}[[Q]]$-valued even symplectic form

$$\Omega(f,g) := \mathrm{Res}_{z=0}(f(-z), g(z))\ dz,$$

where $(\cdot,\cdot)$ is the Poincaré pairing (i.e. $(a,b) = \int_X ab = \langle a, 1, b\rangle_{0,3,0}$). Decomposing $\mathcal{H}$ into the sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ of complementary Lagrangian subspaces (by the standard splitting of a Laurent series into the sum of the power z-series, and the polar part), we identify $\mathcal{H}$ with $T^*\mathcal{H}_+$. Translating the origin in $\mathcal{H}_+$ from 0 to $-1z$ (the operation, refereed to as the *dilatón shift*), we embed the graph of $d\mathcal{F}$ into $\mathcal{H}$ as a Lagrangian submanifold. Explicitly (see [6]):

$$\mathcal{H}_+ \ni t \mapsto \mathcal{J}(t) := -z + t(z) + \sum_{n,d,\alpha} \frac{Q^d}{n!} \phi_\alpha \langle \frac{\phi^\alpha}{-z-\psi}, t(\psi), \dots, t(\psi)\rangle_{0,n,d},$$

where $\{\phi_\alpha\}$ and $\{\phi^\alpha\}$ are Poincaré-dual bases in $H^*(X,\mathbb{Q})$.

In fact, this construction leads to some (rather mild) divergence problem. To elucidate it, pick a graded basis $\{\phi_\alpha\}$ in $H^*(X,\mathbb{Q})$, and assume that $\phi_0 = 1$ and ϕ_α with $\alpha = 1,\dots,r = \mathrm{rk}\, H^2(X)$ are integer degree-2 classes $p_1,\dots,p_r$ taking non-negative values $d_i := p_i(d)$ on degrees $d \in \mathcal{M} \subset H_2(X)$ of holomorphic curves in X. Writing

$$t_k = \sum_\alpha t_{k,\alpha}\phi_\alpha = t_{k,0}1 + t_{k,1}p_1 + \dots + t_{k,r}p_r + \text{the rest of the sum},$$

one can show (on the basis of string and divisor equations), that each Q^d-term in $\mathcal{J}$ contains the factor $e^{t_{0,0}/z}$ (which, unless expanded in powers of $t_{0,0}$, does not fit the space of formal Laurent series in z), and besides comes with the factor $e^{\sum_i d_i t_{0,i}}$ (which is not defined over $\mathbb{Q}$). Also, as it follows from *dilaton equation*, with repect to the variable $t_{1,0}$, the series has convergence radius 1. It follows from dimensional considerations that the rest of each Q^d-term is a polynomial in $1/z$ and in (finitely many of) the coefficients of the power series $t(z)$.

There are several ways to handle the problems. In this paper, we will ignore the convergence properties by interpreting the J-function (and other geometric generating objects) in the sense of formal geometry. That is, $t \mapsto \mathcal{J}(t)$ is cosidered as the germ at $-z$ of a formal series in the components of the vector variables t_k with coefficients which belong to the symplectic loop space.

We will take $\mathbb{Q}[[Q_1, \ldots, Q_r]]$ on the role of the Novikov ring, and represent Q^d by the monomial $Q_1^{d_1} \ldots Q_r^{d_r}$. By virtue of the Q-adic convergence, one can specialize formal variables $t_{k,\alpha}$ to their values in the Novikov ring, taken from its maximal ideal (which is necessary indeed in the case of $t_{0,0}, t_{0,i}$, and $t_{1,0}$). One can also make formal changes of the variables $\{t_{k,\alpha}\}$ with coefficients in the Novikov ring.

Theorem 1. *Let $\sum_d I_d Q^d$, where $I_d(z, z^{-1})$ are cohomology-valued Laurent z-series, represent a point on the graph of $d\mathcal{F}$ in $\mathcal{H}$, and let Φ_α be polynomials in $p_1, \ldots, p_r$ (or, more generally, power z-series, with coefficients polynomial in $p_1, ..., p_r$). Then the family*

$$I(\tau) := \sum_d I_d Q^d \exp\left\{\frac{1}{z}\sum_\alpha \tau_\alpha \Phi_\alpha(p_1 - zd_1, \ldots, p_r - zd_r)\right\}$$

lies on the graph of $d\mathcal{F}$.

Furthermore, for arbitrary scalar power series $c_\alpha(z) = \sum_{k\geq 0} \tau_{\alpha,k} z^k$, the linear combination $\sum_\alpha c_\alpha(z) z \partial_{\tau_\alpha} I$ of the derivatives also lies on the graph.

Moreover, in the case when $p_1, \ldots, p_r$ generate the entire cohomology algebra $H^*(X, \mathbb{Q})$, and Φ_α reprent a linear basis, such linear combinations comprise the whole graph.

`Example 1.` Take $X = \mathbb{C}P^{n-1}$, p to denote the hyperplane class (so that $p^n = 0$), and $\Phi_i = p^i, i = 0, ..., n-1$, for a basis in $\check{H} = H^*(X)$. The "small J-function"

$$(-z)\sum_{d\geq 0} \frac{Q^d}{(p-z)^n (p-2z)^n \cdots (p-dz)^n}$$

is known (see, for instance, [4]) to represent a point on the graph of $d\mathcal{F}$. It follows from the above that the whole graph is comprized by

$$(-z)\sum_{d\geq 0} \frac{Q^d e^{(\tau_0 + \tau_1(p-dz)+\cdots+\tau_{n-1}(p-dz)^{n-1})/z} \sum_{i=0}^{n-1} c_i(z)(p-dz)^i}{(p-z)^n (p-2z)^n \cdots (p-dz)^n},$$

when $c_i(z)$ run arbitrary power series. More explicitly, one equates the power z-series part of this formula to $-z + t(z)$:

$$\sum_{i=0}^{n-1} (\tau_i - z c_i(z)) p^i + (Q\text{-adically small terms}) = -z + \sum_{i=0}^{n-1} p^i \sum_{k=0}^{\infty} t_{k,i} z^k,$$

and expresses τ_i and all coefficients of the series c_i (here $c_0(0)$ needs to lie in a formal neighborhood of 1) in terms of $\{t_{k,i}\}$. Substituting these expressions back into the formula, one obtains (according to Theorem 1) the standard form of the J-function for $\mathbb{C}P^{n-1}$.

In K-theoretic version of GW-theory of a compact Kähler manifold X, the genus-0 descendant potential $\mathcal{F}^K$ is defined by the same formula as its cohomological counterpart:

$$\mathcal{F}^K(t) = \sum_{d\in\mathcal{M}} \sum_{n=0}^{\infty} \frac{Q^d}{n!} \langle t(L, L^{-1}), \dots, t(L, L^{-1})\rangle^K_{0,n,d},$$

using the correlators

$$\langle \Phi_1 L^{k_1}, \dots, \Phi_k L^{k_n}\rangle^K_{0,n,d} := \chi(X_{0,n,d}; \mathcal{O}^{virt} \otimes \mathrm{ev}_1^*(\Phi_1) L_1^{k_1} \cdots \mathrm{ev}_n^*(\Phi_n) L_n^{k_n}).$$

Here χ is the holomorphic Euler characteristic (on $X_{0,n,d}$), $\mathcal{O}^{virt}$ the virtual structure sheaf introduced by Yuan-Pin Lee [11], $\Phi_i \in K^0(X)$ a holomorphic vector bundle on X, $L_i^{k_i}$, $k_i \in \mathbb{Z}$, the k_ith tensor power of the line bundle formed by the cotangent lines to the curves at the ith marked point. The input t in $\mathcal{F}^K$ is a Laurent polynomial of L with coefficients in the K-ring of X.

Adapting the symplectic loop space formalizm, we embed the graph of $d\mathcal{F}^K$ as a Lagrangian submanifold into the "space" $\mathcal{K}$ consisting of power Q-series whose coefficients are rational functions in one indeterminate, q, which take vector values in $K^0(X) \otimes \mathbb{Q}$. Each rational function of q is uniquely writtem as the sum of a Laurent polynomial and a rational function having no pole at $q = 0$ and vanishing at $q = \infty$. The space $\mathcal{K}$ is thereby decomposed into the direct sum of two subspaces, $\mathcal{K}_+$ and $\mathcal{K}_-$ respectively. They are Lagrangian with respect to the symplectic form

$$\Omega^K(f, g) = [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}] \ \ (f(q), g(q^{-1})^K \frac{dq}{q},$$

where $(\cdot,\cdot)^K$ stands for the K-theoretic Poincaré pairing:

$$(A, B)^K = \chi(X; A \otimes B) = \int_X \mathrm{ch}(A)\,\mathrm{ch}(B)\,\mathrm{td}(T_X).$$

Using this Lagrangian polarization to identify $\mathcal{K}$ with $T^*\mathcal{K}_+$, and applying the dilaton shift $1 - q$, we identify the grapf of $d\mathcal{F}^K$ with a submanifold in $\mathcal{K}$, which is described explicitly as follows:

$$t \mapsto 1-q+t(q, q^{-1})+\sum_{n,d,\alpha} \frac{Q^d}{n!} \Phi_\alpha \langle \frac{\Phi^\alpha}{1-qL}, t(L, L^{-1}), \dots, t(L, L^{-1})\rangle^K_{0,n+1,d}.$$

Here Φ_α and Φ^α run Poincaré-dual bases of $K^0(X)$. Similar to the cohomological case, we consider $\mathcal{J}^K$ as a germ (at $1 - q$) of a formal section of $T^*\mathcal{H}_+$. That is, it is a formal series of the coordinates $t_{k,\alpha}$ (on the space of vector Laurent polynomilas $\sum_{k,\alpha} t_{k,\alpha}\Phi_\alpha q^k$), whose coefficients are Q-series with coefficients in rational functions of q.

Let $P_1, ..., P_r$ be line bundles over X such that $c_1(P_i) = -p_i$, i.e. $d_i = -\int_d c_1(P_i)$.

Theorem 2. *Let $\sum_d I_d Q^d$ be a point in $\mathcal{K}$, lying on the graph of $d\mathcal{F}^K$, and let Ψ_α be polynomials in $P_1, \dots, P_r$ (with coeffieints which could be Laurent polynomials in q). Then the family*

$$I^K(\tau) = \sum_d I_d Q^d \exp\{\frac{1}{1-q} \sum_\alpha \tau_\alpha \Psi_\alpha(P_1 q^{d_1}, \dots, P_r q^{d_r})\}$$

also lies on the graph.

Furthermore, for arbitrary scalar Laurent polynomials $c_\alpha(q, q^{-1})$, the linear combinations $\sum_\alpha (1-q)\partial_{\tau_\alpha} I^K$ of the derivatives also lie on the graph.

Moreover, in the case when $P_1, \dots, P_r$ generate the algebra $K^0(X) \otimes \mathbb{Q}$, and Φ_α form a linear basis in it, such linear combinations comprise the whole graph.

`Example 2.` Let X be $\mathbb{C}P^{n-1}$, $P = \mathcal{O}(-1)$ (so that $(1-P)^n = 0$), and $1, 1-P, \dots, (1-P)^{n-1}$ be the basis in $K^0(X)$. It was shown in [7] that the following series lies on the graph of $d\mathcal{F}^K$:

$$(1-q) \sum_{d=0}^{\infty} \frac{Q^d}{(1-Pq)^n (1-Pq^2)^n \cdots (1-Pq^d)^n}.$$

It follows that the whole graph can be parameterized this way:

$$(1-q) \sum_{d=0}^{\infty} \frac{Q^d \; e^{\sum_{i=0}^{n-1} \tau_i (1-Pq^d)^i)/(1-q)} \sum_{i=0}^{n-1} c_i(q, q^{-1})(1-Pq^d)^i.}{(1-Pq)^n (1-Pq^2)^n \cdots (1-Pq^d)^n}.$$

More explicitly, one equates the Laurent polynomial part of this formula to $(1-q) + t(q, q^{-1})$:

$$\sum_{i=0}^{n-1} (1-P)^i (\tau_i + (1-q) c_i(q, q^{-1})) + \mathcal{O}(Q) = 1 - q + \sum_{k,i} t_{k,i} q^k (1-P)^i$$

to express τ_i and all coefficients of the Laurent polynomials c_i in terms of the variables $\{t_{k,i}\}$. Substituting these expressions back into the formula, one obtains the K-theoretic J-function of $\mathbb{C}P^{n-1}$.

`Remark.` For target spaces, whose 2nd cohomology multiplicatively generate the entire cohomology algebra, their cohomological and K-theoretic genus-0 GW-invariants are reconstructible from small degree data, as it is established by the reconstruction results of Kontsevich–Manin [10], Lee–Pandharipande [12], and Iritani–Milanov–Tonita [9]. Our results are closely related to them, and in a sense, explicize the reconstruction procedure.

2. Proof of Theorem 1

Denote by $\mathcal{L} \subset \mathcal{H}$ the graph of $d\mathcal{F}$.

`Step 1.` We begin by noting that modulo Novikov's variables, the graph is known to have the form [6]

$$e^{-\tau/z} z\mathcal{H}_+,$$

where $\tau = \sum_\alpha \tau_\alpha \phi_\alpha$ runs the cohomology space of X.

`Step 2.` The actual graph $\mathcal{L}$ is known (see Appendix 2 in [2]) to have the form

$$S_\tau^{-1}(z) z\mathcal{H}_+,$$

where $\tau \mapsto S_\tau(z)$ is a certain family of matrices (whose entries also depend on Novikov's variables), which has the following properties. Firstly, it is an $1/z$-series: $S = I + \mathcal{O}(1/z)$. Secondly, it belongs to the "twisted" loop group: $S^{-1}(z) = S^*(-z)$, where "$*$" denotes transposition with respect to the Poincaré pairing. Thirdly, it is a fundamental solution to Dubrovin's connection on the tangent bundle of the cohomology space of X:

$$z\partial_\alpha S = \phi_\alpha \bullet S,$$

where $\partial_\alpha := \partial/\partial\tau_\alpha$, and $\phi_\alpha \bullet$ is the matrix of quantum multiplication by ϕ_α (it depends on the application point τ and on Novikov's variables, but not on z, and is self-ajoint). Finally S is constrained by the *string* and *divisor* equations. Namely, assuming as before, that $\{\phi_\alpha\}$ is a graded basis in cohomology, with $\phi_0 = 1$ and $\phi_1 = p_1, \dots, \phi_r = p_r$, we have:

$$z\partial_0 S = S, \quad \text{and} \quad z\partial_i S = zQ_i \partial_{Q_i} S + Sp_i, \quad i = 1, \dots, r.$$

(Here p_i means the operator of multiplication by p_i in the classical cohomology algebra of X.)

Moreover, according to the "descendant–ancestor correspondence" theorem, $S_\tau \mathcal{L}$ is tangent to $\mathcal{H}_+$ along $z\mathcal{H}_+$. This shows that $\mathcal{L}$ is an *overruled Lagrangian cone*. By definition, this means that tangent spaces T_τ to $\mathcal{L}$ (which are $S_\tau^{-1}\mathcal{H}_+$) are tangent to $\mathcal{L}$ exactly along zT_τ.

Lemma. *Let $h \in z\mathcal{H}_+$ so that $f := S_\tau^{-1} h \in \mathcal{L}$, and let $\Phi(\dots, \phi_\alpha, \dots)$ be a polynomial expression of $\{\phi_\alpha\}$. Then the family $e^{\epsilon\Phi(\dots, z\partial_\alpha, \dots)/z} f$ lies in $\mathcal{L}$. The same remains true even if Φ is a power z-series with coefficients polynomial in $\{\phi_\alpha\}$.*

`Proof.` This is a rephrasing of Lemma from the proof of Quantum Lefschetz Theorem in Section 8 of [2]. Here is a variant of the proof.

We have:

$$z\partial_\alpha f = S_\tau^{-1} h = -S_\tau^{-1}(\phi_\alpha \bullet) h \in S_\tau z\mathcal{H}_+.$$

Even when vector $h \in z\mathcal{H}_+$ depends on τ, Leibnitz' rule shows that $z\partial_\alpha f \in S_\tau z\mathcal{H}_+$. Therefore $\Phi(\dots, z\partial_\alpha, \dots) f \in S_\tau^{-1} z\mathcal{H}_+$, and by the string equation, after division by z, still lies in the tangent space to $\mathcal{L}$ at the point f. Integrating this vector field on $\mathcal{L}$ during time ϵ (which is certainly possible at least in the ϵ-adic sense), we obtain the result.

Note that if $\mathcal{O}(z)$-terms are added to Φ, the conclusion still holds (by the same arguments, or even simpler: because the ruling spaces $zT_\tau \subset \mathcal{L}$ are $\mathbb{Q}[[z]]$-modules).

`Step 3.` Let $\mathcal{D}$ be the algebra of differential operators in Novikov's variables. It follows from the above divisor equations for S that *tangent spaces* $T_\tau = S_\tau^{-1}\mathcal{H}_+$ *to* $\mathcal{L}$ *are* $\mathcal{D}$*-modules* with respect to the action of $\mathcal{D}$ defined by the multiplication operators Q_j and differentiation operators $zQ_i\partial_{Q_i} - p_i$, where p_i stands for multiplication by p_i in the classical cohomology algebra of X. The same is true about the *ruling spaces* $z\mathcal{T}_\tau$. It follows that, if Φ is a polynomial expression in $zQ_i\partial_{Q_i} - p_i$, and T is the tangent space to $\mathcal{L}$ at $f \in zT$, then $\Phi f/z \in T$, and therefore the flow $f \mapsto e^{\epsilon\Phi/z} f$ preserves $\mathcal{L}$.

`Remark.` Since we are using differentiations in Q, it seems counter-intuitive to think of Novikov's variables as contants. In fact one can think of the symplectic loop space $\mathcal{H}$ geometrically as the space of formal sections over the spectrum of the Novikov ring, of the bundle whose fiber consists of Laurent z-series with vector coefficients. Likewise, the cone $\mathcal{L} \subset \mathcal{H}$ consists of sections of the fibration whose fibers are overruled Lagrangian cones. The differential operators Φ/z and their flows $e^{\epsilon\Phi/z}$ act by linear transformations on the space of section $\mathcal{H}$. Thus, $f \in \mathcal{L}$ is a section

$$Q \mapsto f(Q) = S_{\tau(Q)}^{-1}(z^{-1}, Q) z h(z, Q),$$

where $h(\cdot, Q)$ is a given power z-series. Then straightforward differentiation together with the divisor equation for S shows that $(zQ_i\partial_{Q_i} - p_i) f$ lies in zT, where T is the tangent space to $\mathcal{L}$ at f. This implies that for any expression Φ which is a z-series with coefficients polynomial in $zQ_i\partial_{Q_i} - p_i$, $\Phi f/z$ still lies in T, and hence the flow $e^{\epsilon\Phi/z}$ preserves $\mathcal{L}$. In other words, $e^{\epsilon\Phi/z} f$ is an ϵ-family of sections $Q \mapsto f(\epsilon, Q)$ of the fibration of overruled Lagrangian cones. One can choose any function $Q \mapsto \epsilon(Q)$ to obtain the section $Q \mapsto f(\epsilon(Q), Q)$ lying in $\mathcal{L}$. We should note that it differs from $e^{\epsilon(Q)\Phi/z} f$ since multiplication by $\epsilon(Q)$ and Φ do not commute.

`Step 4.` Write $f=\sum_d f_dQ^d$. Then

$$e^{\epsilon\Phi(\dots,p_i-zQ_i\partial_{Q_p},\dots)/z}f=\sum_d f_dQ^de^{\epsilon\Phi(\dots,p_i-zd_i,\dots)/z},$$

which according to Step 3 lies in $\mathcal{L}$ whenever f does. Here one can consider ϵ as a parameter, or take its value from the Novikov ring (or at least from its maximal ideal).

One obtains the first statement of Theorem 1 by raplacing $\epsilon\Phi$ with a linear combination $\sum\tau_\alpha\Phi_\alpha$ of commuting differential operators.

The derivatives $\partial_\alpha I(\tau)$ lie in the tangent space T to $\mathcal{L}$ at I_τ, and hence all linear combination $\sum c_\alpha(z)z\partial_\alpha I(\tau)$, where c_α are scalar power z-series, lie in the same ruling space $zT\subset\mathcal{L}$.

When $p_1,\dots,p_r$ generate the entire cohomology algebra of X, it follows from Step 1 that *modulo Novikov's variables*, such linear combinations compise the whole of $\mathcal{L}$. Now the formal Implicit Function Theorem implies the last statement of Theorem 1.

3. Proof of Theorem 2

Let $\mathcal{L}^K\subset\mathcal{K}$ denote the graph of $d\mathcal{F}^K$.

It is known (as explained in [8], Section 3) that $\mathcal{L}^K$ is an overruled Lagrangian cone too. More precisely, as in the case of quantum cohomology theory, there is a family $\tau\mapsto S_\tau(q,Q)$ of matrices depending on $\tau\in K^*(X)$ which transform $\mathcal{L}^K$ to $S_\tau\mathcal{L}$ tangent to $\mathcal{K}_+$ along $(1-q)\mathcal{K}_+$. As a consequence, $\mathcal{L}^K$ is a cone whose tangent spaces $T_\tau=S_\tau^{-1}\mathcal{K}_+$ are $\mathbb{Q}[q,q^{-1}]$ modules, and are tangent to $\mathcal{L}^K$ exactly along $(1-q)T_\tau$.

The main result of [8] is the Quantum Hirzebruch-Riemann–Roch Theorem which completely characterizes $\mathcal{L}^K$ in terms of $\mathcal{L}$ in the following, "adelic" way. For each complex value ζ of $q\neq 0$, one introduces the localization space $\mathcal{K}^\zeta$ which consists of series in Q whose coefficients are vectors in $K^0(X)\otimes\mathbb{Q}(\zeta)$ and formal Laurent series in $1-q\zeta$. One equips it with the symplectic form

$$\Omega^\zeta(f,g):=\mathrm{Res}_{q=\zeta^{-1}}(f(q^{-1},g(q))^K\,\frac{dq}{q}.$$

It is designed so that the adelic map

$$(\mathcal{K},\Omega)\to\widehat{\mathcal{K}},\widehat{\Omega}:=\prod_\zeta(\mathcal{K}^\zeta,\Omega^\zeta),$$

which is defined by assigning to a rational function f of q the collection (f^ζ) of its Laurent series expansions (one at each $q=\zeta^{\mathbf{1}}$), preserves the symplectic form:

$$\widehat{\Omega}(\widehat{f},\widehat{g}) := \sum_{\zeta} \Omega^{\zeta}(f^{\zeta}, g^{\zeta}) = \Omega(f,g).$$

Decomposing Laurent series into the power series part and the polar part, one obtains lagrangian polarizations $\mathcal{K}^{\zeta} = \mathcal{K}^{\zeta}_{+} \oplus \mathcal{K}^{\zeta}_{-}$, identifying each $(\mathcal{K}^{\zeta}, \Omega^{\zeta})$ with $T^{*}\mathcal{K}^{\zeta}_{+}$.

Next, in each $(\mathcal{K}^{\zeta}, \Omega^{\zeta})$, a certain Lagrangian submanifold $\mathcal{L}^{\zeta}$ is described. For ζ which is not a root of 1, $\mathcal{L}^{\zeta} = \mathcal{K}^{\zeta}_{+}$. For $\zeta = 1$, $\mathcal{L}^{1} \subset \mathcal{K}^{1}$ is the graph of the differential of $\mathcal{F}^{fake}$, the genus-0 descendant potential of *fake* quantum K-theory (studied in [6, 3, 1]). For $\zeta \neq 1$, which is a primitive mth root of 1, $\mathcal{L}^{\zeta}$ is a certain linear subspace originating in a certain fake twisted quantum K-theory with the orbifold target space $X/\mathbb{Z}_m$ (see [8] for more detail).

The adelic characterization of $\mathcal{L}^{K}$ says that $f \in \mathcal{L}^{K}$ *if and only if* $f^{\zeta} \in \mathcal{L}^{\zeta}$ *for each* ζ.

Furthermore, $\mathcal{L}^{\zeta}$ have the following description in terms of the cone $\mathcal{L} \subset \mathcal{H}$ of quantum cohomology theory.

First, the *quantum Chern character* defines an isomorphism qch : $\mathcal{K}^{1} \to \mathcal{H}$. By definition, qch acts by the usual Chern character on the coefficients of Laurent $q-1$-series, preserves Novikov's variables, and transforms q into e^{z}. According to the "quantum HRR theorem" in *fake* quantum K-theory [1, 3],

$$\mathcal{L}^{1} = \mathrm{qch}^{-1} \triangle \mathcal{L}, \quad \text{where} \;\; \triangle \sim \prod_{\text{Chern roots } x \text{ of } T_X} \prod_{r=1}^{\infty} \frac{x - rz}{1 - e^{-x+rz}}.$$

Here $\sim$ means taking the "Euler-Maclaurin asumptotics" of the R.H.S. We won't remind the reader what it is (see, for instance, [2, 8]), but note that (just as the expression on the R.H.S. suggests), it is multiplication by a series in $z^{\pm 1}$ built of operators of multiplication in classical cohomology algebra of X, but *independent of Novikov's variables.*

Then, when $\zeta \neq 1$ is a primitive mth root of 1, one can give the following (comewhat clumsy) cohomological description of $\mathcal{L}^{\zeta}$. On the cone $\mathcal{L}^{K}$, there is the point, denoted $\mathcal{J}(0)$ which corresponds to the input $t = 0$. It is called the "small J-function," and modulo $\mathcal{K}_{-}$, it is congruent to the dilaton shift $1 - q$. The tangent space to $\mathcal{L}^{K}$ at this point is $T_0 := S_0^{-1}(q, Q)\mathcal{K}_{+}$. Expanding $\mathcal{J}(0)$ into a Laurent series in $q - 1$, we obtain the corresponding point $\mathcal{J}(0)^{1}$ in $\mathcal{L}^{fake}$, and doing the same with the matrix S_0 obtain the tangent space to $\mathcal{L}^{fake}$ at $\mathcal{J}(0)^{1}$: $T_0^{1} = (S_0^{-1})^{1}\mathcal{K}^{1}_{+}$. Replacing in this matrix q with q^{m}, and all Q_i with Q_i^{m}, one introduces the subspace $\mathcal{T}^{\zeta} := (S_0^{-1}(q^{m}, Q^{m}))^{\zeta}\mathcal{K}^{\zeta}_{+}$ in $\mathcal{K}^{\zeta}$.

Finally, (see [8] for further details),

$$\mathcal{L}^\zeta = \nabla_\zeta \mathcal{T}^\zeta, \quad \text{where} \quad \nabla_\zeta \ \sim_{q\zeta\to 1} \prod_{\substack{\textit{K-theoretic Chern} \\ \textit{roots } P \textit{ of } T_X^*}} \frac{\prod_{r=1}^\infty (1-q^{mr}P)}{\prod_{r=1}^\infty (1-q^r P)}.$$

Here "$\sim_{q\zeta\to 1}$" refers to Euler–Maclaurin asymptotizs as $z\to 0$ when $q\zeta$ is replaced with e^z, and the K-theoretic Chern roots are characterized by $\operatorname{ch} P = e^{-x}$, where x runs the Chern roots of T_X.

Let $P_1,\dots,P_r$ be line bundles on X such that $c_1(P_i) = -p_i$, and let $\mathcal{D}_q$ be the algebra of finite-difference operators in Novikov's variables. By definition, it acts on $\mathcal{K}$ by the "translation" operators $P_i q^{Q_i\partial_{Q_i}} = \exp(-p_i + (\log q) Q_i \partial_{Q_i})$ and multiplications by Q_j.

It is shown in [8] that all tangent spaces T_τ to $\mathcal{L}^K$ (as well as the ruling subspaces $(1-q)T_\tau \subset \mathcal{L}^K$ of the cone) are $\mathcal{D}_q$-modules. Here we adapt the argument from [8] to prove Theorem 1.

Namely, the translation operators are exponentials of the differentiation operators from the algebra $\mathcal{D}$ (we assume, of course, that $q = e^z$). As a consequence of the divisor equations, all tangent spaces to $\mathcal{L} \subset \mathcal{H}$ are $\mathcal{D}$-modules, and therefore invariant under the translation operators as well. The same is true about all tangent spaces to the cone $\mathcal{L}^1 \subset \mathcal{K}^1$ because the operator $\triangle$ commutes with $\mathcal{D}$. Likewise, ∇_ζ commutes with $\mathcal{D}$, and so for each $\zeta \neq 1$ the space $\mathcal{L}_\zeta$ is also $\mathcal{D}$- and $\mathcal{D}_q$-invariant (since this is true for $\mathcal{T}^\zeta$, as it is explained in Lemma of Section 9 in [8]). Besides, as we know from the previous section, operators of the form

$$\exp\left[\Phi(\dots, -p_i + (\log q) Q_i \partial_{Q_i}, \dots)/(1-q)\right]$$

preserve $\mathcal{L} \in \mathcal{H}$ (here we assume $q = e^z$ and use the fact that the pole of $1/(1-e^z)$ at $z = 0$ is simple). Since they commute with $\triangle$, they preserve $\mathcal{L}^1 \subset \mathcal{K}^1$.

Thus, the proof of Theorem 2 goes as follows. Take in $\mathcal{D}_q$ a family of operators of the form

$$D := \exp\left[\epsilon \Psi(\dots, P_i q^{Q_i\partial_{Q_i}}, \dots)/(1-q)\right],$$

and apply it to a vector $f \in \mathcal{L}^K$. It is important that the result Df is a family in the space $\mathcal{K}$ of Q-series with coefficients rational in q.

By the adelic characterization of $\mathcal{L}^K$, each localization vector f^ζ lies in $\mathcal{L}^\zeta$. For $\zeta \neq 1$, D acts on $\mathcal{K}^\zeta$ by pseudo-differential operators, which therefore preserve $\mathcal{L}_\zeta$. Thus $(Df)^\zeta$ is a family in $\mathcal{L}^\zeta$. For $\zeta = 1$, the operator D has the form of the flow

$$\exp\left[\epsilon \Psi(\dots, e^{-p_i + (\log q) Q_i \partial_{Q_i}}, \dots)/(1-q)\right],$$

which preserves $\mathcal{L}^1 \subset \mathcal{K}^1$. Thus $(Df)^1$ is a family in $\mathcal{L}^1$. By the adelic characterization of $\mathcal{L}^K$, it follows that Df is a family in $\mathcal{L}^K$.

Decomposing f into Q-series $\sum_d f_d Q^d$, we find that

$$Df = \sum_d f_d Q^d e^{\epsilon\Psi(\dots,P_i q^{d_i},\dots)/(1-q)}.$$

The proof now ends the same way as in Theorem 1. Replacing $\epsilon\Psi$ with a linear combination $\sum \tau_\alpha \Psi_\alpha$ of finite difference operators, one concludes that the family $\tau \mapsto I^K(\tau)$ lies in $\mathcal{L}^K$. The derivatives $\partial_\alpha I(\tau)$ lie in the tangent space T to $\mathcal{L}^K$ at $I^K(\tau)$. Since T is a module over $\mathbb{Q}[q,q^{-1}]$, and $(1-q)T \subset \mathcal{L}^K$, one finds that $\sum c_\alpha(q,q^{-1})(1-q)\partial_\alpha I^K(\tau)$ also lie in $\mathcal{L}^K$. Finally, assuming that $P_1,\dots,P_r$ generate $K^0(X)$, one derives that such linear combinations comprise the whole of $\mathcal{L}^K$ by checking this statement modulo Novikov's variables, and employing the formal Implicit Function Theorem.

4. Further implications and generalizations

A. Birkhoff factorizations and mirror maps. When $H^*(X,\mathbb{Q})$ is generated by the degree-2 classes $p_1,\dots,p_r$, Theorems 1 and 2 can be reformulated as the following reconstruction results for the "S-matrix." Starting with polynomials $\Phi_\alpha(p)$ representing a basis in $H^*(X,\mathbb{Q})$, and with a point $\sum I_d Q^d$ on the cone $\mathcal{L}$, one obtains a family of such points

$$I(\tau) = \sum I_d(z,z^{-1}) Q^d e^{-\sum \tau_\alpha \Phi_\alpha(p-dz)/z}.$$

We may assume here that $I_0 = -z$. The derivatives $\partial_\alpha I$ form a $\mathbb{Q}[[z]]$-basis in the tangent spaces to $\mathcal{L}$ (depending on τ). The square maqtrix $U := [(\partial_\alpha I, \phi^\beta)]$, formed by the components of these derivatives, can be factored into the product of $U(z,z^{-1}) = V(z)W(z^{-1})$ of two matrix series (in the variables τ and Q), whose coefficients are power series of z (on the left) and polynomial functions of z^{-1} (on the right). In the procedure (known as Birkhoff factorization), one may assume that $W(0) = I$. has the form Then W coincides with $S_\tau(-z^{-1})$ up to a change of variables $\tau_\alpha \mapsto \tau_\alpha + \mathcal{O}(Q)$, which generalizes the "mirror map" known in the mirror theory. To describe the change of variables, assume that $\Phi_0 = 1$, and note that the "first row" of W has the form

$$1 - z^{-1} \sum \Phi_\alpha(p)(\tau_\alpha + \mathcal{O}(Q)) + o(z^{-1}).$$

The mirror map is read off the z^{-1}-term of the expansion.

In quantum K-theory, a similar result is obtained by Birkhoff factorization $U = VW$, where the entries of U, V, and W are built respectively of arbitrary rational functions, Laurent polynomials, and reduced rational functions of q reqular at $q = 0$.

B. Torus-equivariant theory. It is often useful [5] to consider GW-invariants *equivariant* with respect to a torus action on X. The above results apply to this case without any significant changes. One only needs to extend the coefficient ring by the power series completion $\mathbb{Q}[[\lambda]]$ of the coefficient ring of the equivariant theory. For example, when the torus T^n of diagonal matrices acts on $X = \mathbb{C}P^{n-1} = \mathrm{proj}(\mathbb{C}^n)$, the T^n-equivariant cohomology algebra of X is described by the relation $(p-\lambda_1)\cdots(p-\lambda_n) = 0$. For the purpose of employing fixed-point localization, it is convenient to assume that the hyperplane class p localizes to each of the values λ_j. However, for the purpose of our proof it suffices to assume that λ_j are generators of the formal series ring $\mathbb{Q}[[\lambda_1, \dots, \lambda_n]]$, and obtain the following parameterization of the graph $d\mathcal{F}$ in the T^n-equivariant GW-theory:

$$(-z)\sum_{d\geq 0} \frac{Q^d e^{(\tau_0+\tau_1(p-dz)+\dots+\tau_{n-1}(p-dz)^{n-1})/z}\sum_{i=0}^{n-1} c_i(z)(p-dz)^i}{\prod_{j=1}^n (p-\lambda_j - z)(p-\lambda_j-2z)\cdots(p-\lambda_j-dz)},$$

where the fractions $1/(p-\lambda-rz)$ are interpreted as Laurent polynomials in z^{-1} modulo high powers of λ.

C. Twisted GW-invariants. Our results also extend to the case of twisted GW-invariants in the sense of [2] (e.g. "local" ones, i.e. GW-invariants of the non-compact total space of a vector bundle $E \to X$, or GW-invarinats of the "super-bundle" $\Pi E \to X$, which in genus 0 are closely related to those of the zero locus of a section of E). In such cases, to remove degenerations caused by non-compactness, one needs to act equivariantly, equipping E with the fiberwise scalar circle action. To adapt our arguments to this case, it suffices to work over the coefficient ring $H^*(BS^1, \mathbb{Q}) = \mathbb{Q}[\lambda]$ localized to $\mathbb{Q}((\lambda))$. For example, the graph of $d\mathcal{F}$ of the local theory on the total space E of degree l line bundle over $\mathbb{C}P^{n-1}$, for $l > 0$ obtains the following description:

$$(-z)\sum_{d\geq 0} \frac{Q^d e^{(\tau_0+\tau_1(p-dz)+\dots+\tau_{n-1}(p-dz)^{n-1})/z}\sum_{i=0}^{n-1} c_i(z)(p-dz)^i}{\prod_{r=0}^{ld}(lp+\lambda-rz)\ \prod_{r=1}^d (p-rz)^n}.$$

Here $p^n = 0$, while the fractions $1/(lp+\lambda-rz)$ should be expanded as power z-series, whose coefficients, however, can be Laurent series of λ.

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June 25, 2014