1.4. Complex numbers

The quadratic equation $x^2 - 1 = 0$ in one unknown has two solutions $x = \pm 1$. The equation $x^2+1 = 0$ has no solutions at all. For the sake of justice one introduces a new number i, the imaginary unit, such that $i^2 = -1$, and thus $x = \pm i$ become two solutions to the equation.

1.4.1. Definitions and geometrical interpretations. Complex numbers are defined as ordered pairs of real numbers written in the form $z = a + bi$. The real numbers a and b are called the real part and imaginary part of the complex number z and denoted $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. The sum of two complex numbers z and $w = c + di$ is defined by $z + w = (a + c) + (b + d)i$ while the definition of the product is to comply with the relation $i^2 = -1$:

$$zw = ac + bdi^2 + adi + bci = (ac - bd) + (ad + bc)i.$$

Operations of addition and multiplication of complex numbers enjoy the same properties as those of real numbers. In particular, the product is commutative and associative.

The complex number $\bar{z} = a - bi$ is called complex conjugate to $z = a + bi$. The formula $\overline{z + w} = \bar{z} + \bar{w}$ is obvious, and $\overline{zw} = \bar{z}\bar{w}$ is due to the fact that $\bar{i} = -i$ has exactly the same property as i: $(-i)^2 = -1$.

The product $z\bar{z} = a^2 + b^2$ (check this formula!) is real and positive unless $z = 0 + 0i = 0$. This shows that

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i,$$

and hence the division by z is well-defined for any non-zero complex number z.

The non-negative real number $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ is called the absolute value of z. The absolute value function has the same multiplicative property as in the case of real numbers: $|zw| = \sqrt{zw\overline{zw}} = \sqrt{z\bar{z}w\bar{w}} = |z| \cdot |w|$. It actually coincides with the absolute value of real numbers when applied to complex numbers with zero imaginary part: $|a + 0i| = |a|$.

To a complex number $z = a+bi$, we can associate the radius-vector $\mathbf{z} = a\mathbf{e}_1+b\mathbf{e}_2$ on the coordinate plane. The unit coordinate vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ represent therefore the complex numbers 1 and i. The coordinate axes are called respectively the real and imaginary axes of the plane. Addition of complex numbers coincides with the operation of vector sum.

The absolute value function has the geometrical meaning of the distance to the origin: $|z| = \langle \mathbf{z}, \mathbf{z} \rangle^{1/2}$, while $z\bar{z}$ is the inner square. In particular, the triangle inequality $|z + w| \le |z| + |w|$ holds true. Complex numbers of unit absolute value $|z| = 1$ form the unit circle centered at the origin.

The operation of complex conjugation acts on the vectors $\mathbf{z}$ as the reflection about the real axis.

In order to describe a geometrical meaning of complex multiplication, let us write the vector representing a non-zero complex number z in the polar (or trigonometric) form $z = ru$ where $r = |z|$ is a positive real number, and $u = z/|z| = \cos\theta + i \sin\theta$ has the absolute value 1. Here $\theta = \arg z$, called the argument of the complex number z, is the angle that the vector $\mathbf{z}$ makes with the positive direction of the real axis.

Consider the linear transformation on the plane defined as multiplication of all complex numbers by a given complex number z. It is the composition of the multiplication by r and by u. The geometrical meaning of multiplication by r is clear: it makes all vectors r times longer. The multiplication by u is described by the following formulas

$$\begin{array}{l}\mathrm{Re}\,[(\cos\theta + i\sin\theta)(x_1 + ix_2)] = (\cos\theta)x_1 - (\sin\theta)x_2 \\ \mathrm{Im}\,[(\cos\theta + i\sin\theta)(x_1 + ix_2)] = (\sin\theta)x_1 + (\cos\theta)x_2\end{array}.$$

This is a linear transformation with the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. The multiplication by u is therefore the rotation through the angle θ. Thus the multiplication by z is the composition of the dilation by $|z|$ and rotation through $\arg z$.

In other words, the product operation of complex numbers sums their arguments and multiplies absolute values:

$$|zw| = |z|\cdot|w|, \quad \arg zw = \arg z + \arg w \text{ modulo } 2\pi.$$

1.4.2. The exponential function. Consider the series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots + \frac{z^n}{n!} + \dots$$

Applying the ratio test for convergence of infinite series,

$$\left|\frac{z^n(n-1)!}{n!z^{n-1}}\right| = \frac{|z|}{n} \to 0 < 1 \text{ as } n \to \infty,$$

we conclude that the series converges absolutely for any complex number z. The sum of the series is a complex number denoted $\exp z$, and the rule $z \mapsto \exp z$ defines the exponential function of the complex variable z.

The exponential function transforms sums to products:

$$\exp(z + w) = (\exp z)(\exp w) \text{ for any complex } z \text{ and } w.$$

Indeed, due to the binomial formula, we have

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = n! \sum_{k+l=n} \frac{z^k}{k!}\frac{w^l}{l!}.$$

Rearranging the sum over all n as a double sum over k and l we get

$$\sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{z^k}{k!}\frac{w^l}{l!} = (\sum_{k=0}^{\infty} \frac{z^k}{k!})(\sum_{l=0}^{\infty} \frac{w^l}{l!}).$$

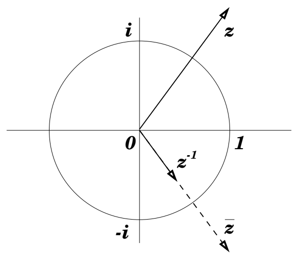
The exponentials of complex conjugated numbers are conjugated:

$$\exp \bar{z} = \sum \frac{\bar{z}^n}{n!} = \overline{\sum \frac{z^n}{n!}} = \overline{\exp z}.$$

In particular, on the real axis the exponential function is real and coincides with the usual real exponential function $\exp x = e^x$ where $e = 1 + 1/2 + 1/6 + \dots + 1/n! + \dots = \exp(1)$. Extending this notation to complex numbers we can rewrite the above properties of $e^z = \exp z$ as $e^{z+w} = e^z e^w$, $e^{\bar{z}} = \overline{e^z}$.

On the imaginary axis, $w = e^{iy}$ satisfies $w\bar{w} = e^0 = 1$ and hence $|e^{iy}| = 1$. The way the imaginary axis is mapped by the exponential function to the unit circle is described by the following Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$



Exercises 1.4.1.

(a) Compute $(1+i)/(3-2i)$, $(\cos\pi/3 + i\sin\pi/3)^{-1}$.

(b) Show that z^{-1} is a real proportional to $\bar{z}$ and find the proportionality coefficient.

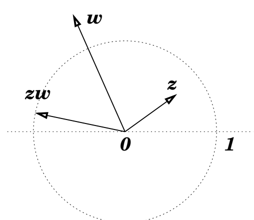
(c) Find all z satisfying $|z-1| = |z-2| = 1$.

(d) Sketch the solution set to the following system of inequalities: $|z+1| \le 1$, $|z| \le 1$, $\mathrm{Re}(iz) \le 0$.

(e) Compute $(\frac{\sqrt{3}+i}{2})^{100}$.

(f) Prove that the linear transformation defined by the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is the composition of multiplication by $\sqrt{a^2+b^2}$ and a rotation.

(g) Let $z_1, ..., z_5$ form a regular pentagon inscribed into the unit circle $|z| = 1$. Prove that $z_1 + ... + z_5 = 0$.



Exercises 1.4.2.

(a) Prove the "Fundamental Formula of Mathematics": $e^{\pi i} + 1 = 0$.

(b) Represent $1-i$ and $1-\sqrt{3}i$ in the polar form $re^{i\theta}$.

(c) Show that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$.

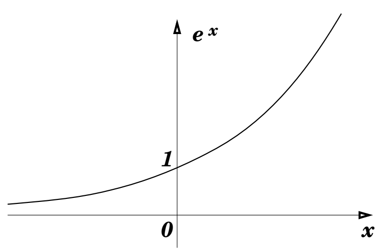
(d) Compute the real and imaginary part of the product $e^{i\phi}e^{i\psi}$ using the Euler formula and deduce the addition formulas for $\cos(\phi+\psi)$ and $\sin(\phi+\psi)$.

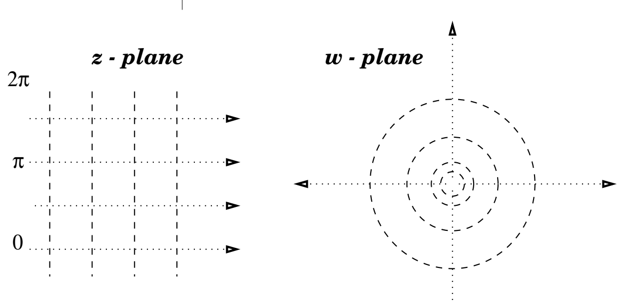
(e) Express $\mathrm{Re}\, e^{3i\theta}$, $\mathrm{Im}\, e^{3i\theta}$ in terms of $\mathrm{Re}\, e^{i\theta}$ and $\mathrm{Im}\, e^{i\theta}$ and deduce the triple argument formulas for $\cos 3\theta$ and $\sin 3\theta$.

(f) Prove the binomial formula:

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k},$$

where $\binom{n}{k} = n!/k!(n-k)!$.





w = exp (z)

It is proved by comparison of $e^{i\theta} = \sum (i\theta)^n/n!$ with Taylor series for $\cos\theta$ and $\sin\theta$:

$$\begin{array}{l} \operatorname{Re} e^{i\theta} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - ... = \sum (-1)^k \frac{\theta^{2k}}{(2k)!} = \cos\theta \\ \operatorname{Im} e^{i\theta} = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - ... = \sum (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} = \sin\theta \end{array}.$$

Thus $\theta \mapsto e^{i\theta}$ is the usual parameterization of the unit circle by the angular coordinate θ. In particular, $e^{2\pi i} = 1$ and therefore the exponential function is $2\pi i$-periodic: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$. Using Euler's formula we can rewrite the polar form of a non-zero complex number w as

$$w = |w| e^{i \arg w}.$$

1.4.3. The Fundamental Theorem of Algebra. A quadratic polynomial $z^2 + pz + q$ has two roots

$$z_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

regardless of the sign of the discriminant $p^2 - 4q$, if we allow the roots to be complex and take in account multiplicity. Namely, if $p^2 - 4q = 0$, $z^2 + pz + q = (z + p/2)^2$ and therefore the single root $z = -p/2$ has multiplicity two. If $p^2 - 4q < 0$ the roots are complex conjugated with $\operatorname{Re} z_\pm = -p/2$, $\operatorname{Im} z_\pm = \pm\sqrt{|p^2 - 4q|}/2$. The Fundamental Theorem of Algebra shows that not only the justice has been restored, but that any degree n polynomial has n complex roots, possibly — multiple.

Theorem. *A degree n polynomial $P(z) = z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$ with complex coefficients $a_1, ..., a_n$ factors as*

$$P(z) = (z - z_1)^{m_1} ... (z - z_r)^{m_r}.$$

Here $z_1, ..., z_r$ are complex roots of $P(z)$ of multiplicities $m_1, ..., m_r$, and $m_1 + ... + m_r = n$.

This is one of a few theorems we intend to use in this course without proof. We illustrate it with the following examples.

Examples. (a) The equation $z^2 = w$, where $w = re^{i\theta}$ is a complex number written in the polar form, has two solutions $\pm\sqrt{w} = \pm\sqrt{r} e^{i\theta/2}$. Thus the formula for roots of quadratic polynomials makes sense even if the coefficients p, q are complex.

(b) The complex numbers $1, i, -1, -i$ are the roots of the polynomial $z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i)$.

(c) There are n complex n-th roots of unity. Namely, if $z = re^{i\theta}$ satisfies $z^n = 1$ then $r^n e^{in\theta} = 1$ and therefore $r = 1$ and $n\theta = 2\pi k$, $k = 0, \pm 1, \pm 2, ...$ Thus

$$z = e^{2\pi i k/n} = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n},\ k = 0, 1, 2, ..., n - 1.$$

For instance, if $n = 3$, the roots are 1 and

$$e^{\pm 2\pi i/3} = \cos\frac{2\pi}{3} \pm i \sin\frac{2\pi}{3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

As illustrated by the previous two examples, if the coefficients $a_1, ..., a_n$ of the polynomial $P(z)$ are real numbers, that is $\bar{a}_i = a_i$, yet the roots can be non-real, but then they come in complex conjugated pairs. This follows from equality of two factorizations for $\overline{P(\bar{z})} = z^n + \bar{a}_1 z^{n-1} + ... + \bar{a}_n = P(z)$:

$$(z - \bar{z}_1)^{m_1} ... (z - \bar{z}_r)^{m_r} = (z - z_1)^{m_1} ... (z - z_r)^{m_r}.$$

These equal products can differ only by the order of the factors, and thus for each non-real root of $P(z)$ the complex conjugate is also a root and of the same multiplicity.

Expanding the product

$$(z - z_1)...(z - z_n) = z^n - (z_1 + ... + z_n)z^{n-1} + ... + (-1)^n z_1...z_n$$

we can express coefficients $a_1, ..., a_n$ of the polynomial via the roots $z_1, ..., z_n$ (here multiple roots should be repeated according to their multiplicities). In particular, the sum and the product of roots are

$$z_1 + ... + z_n = -a_1, \quad z_1...z_n = (-1)^n a_n.$$

These formulas generalize the Vieta theorem for roots of quadratic polynomials: $z_+ + z_- = -p,\ z_+ z_- = q$.

Exercises 1.4.3.

(a) Solve the quadratic equations:

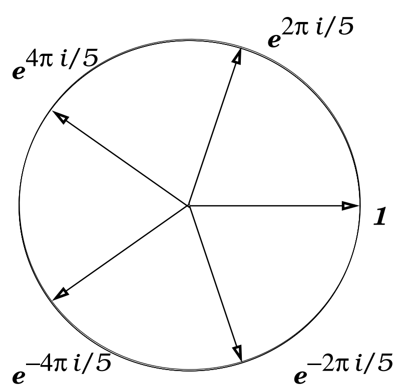
$$z^2 - 6z + 5 = 0, \quad z^2 - iz + 1 = 0, \quad z^2 - 2(1+i)z + 2i = 0, \quad z^2 - 2z + \sqrt{3}i = 0.$$

(b) Solve the equations

$$z^3 + 8 = 0, \quad z^3 + i = 0, \quad z^4 + 4z^2 + 4 = 0, \quad z^4 - 2z^2 + 4 = 0, \quad z^6 + 1 = 0.$$

(c) Prove that for any $n > 1$ the sum of all n-th roots of unity equals zero.

(d) Prove that any polynomial with real coefficients factors into a product of linear and quadratic polynomials with real coefficients.



5-th roots of unity