

## 3 Complex Numbers

### Law and Order

Life is unfair: The quadratic equation  $x^2 - 1 = 0$  has two solutions  $x = \pm 1$ , but a similar equation  $x^2 + 1 = 0$  has no solutions at all. To restore justice one introduces new number  $i$ , the **imaginary unit**, such that  $i^2 = -1$ , and thus  $x = \pm i$  become two solutions to the equation. This is how complex numbers could have been invented.

More formally, complex numbers are introduced as ordered pairs  $(a, b)$  of real numbers, written in the form  $z = a + bi$ . The real numbers  $a$  and  $b$  are called respectively the **real part** and **imaginary part** of the complex number  $z$ , and are denoted  $a = \operatorname{Re} z$  and  $b = \operatorname{Im} z$ .

The sum of  $z = a + bi$  and  $w = c + di$  is defined as

$$z + w = (a + c) + (b + d)i.$$

The product is defined so as to comply with the relation  $i^2 = -1$ :

$$zw = ac + bdi^2 + adi + bci = (ac - bd) + (ad + bc)i.$$

The operations of addition and multiplication of complex numbers enjoy the same properties as those of real numbers do. In particular, the product is commutative and associative.

The complex number  $\bar{z} = a - bi$  is called **complex conjugate** to  $z = a + bi$ . The operation of complex conjugation *respects* sums and products:

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

This can be easily checked from definitions, but there is a more profound explanation. The equation  $x^2 + 1 = 0$  has two roots,  $i$  and  $-i$ , and the choice of the one to be called  $i$  is totally ambiguous. The complex conjugation consists in systematic renaming  $i$  by  $-i$  and *vice versa*, and such renaming cannot affect properties of complex numbers.

Complex numbers satisfying  $\bar{z} = z$  are exactly the real numbers  $a + 0i$ . We will see that this point of view on real numbers as complex numbers *invariant* under complex conjugation is quite fruitful.

The product  $z\bar{z} = a^2 + b^2$  (check this formula!) is real, and is positive unless  $z = 0 + 0i = 0$ . This shows that

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

Hence the division by  $z$  is well-defined for any non-zero complex number  $z$ . In terminology of Abstract Algebra, complex numbers form therefore a **field**<sup>6</sup> (just as real or rational numbers do).

The field of complex numbers is denoted by  $\mathbb{C}$  (while  $\mathbb{R}$  stands for reals, and  $\mathbb{Q}$  for rationals).

The non-negative real number  $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$  is called the **absolute value** of  $z$ . The absolute value function is **multiplicative**:

$$|zw| = \sqrt{zw\bar{z}\bar{w}} = \sqrt{z\bar{z}w\bar{w}} = |z| \cdot |w|.$$

It actually coincides with the absolute value of real numbers when applied to complex numbers with zero imaginary part:  $|a + 0i| = |a|$ .

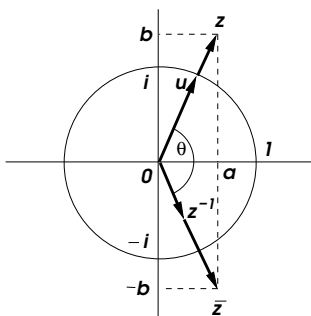


Figure 17

## Geometry

We can identify complex numbers  $z = a + bi$  with points  $(a, b)$  on the real coordinate plane (Figure 17). This way, the number 0 is identified with the origin, and 1 and  $i$  become the unit basis vectors  $(1, 0)$  and  $(0, 1)$ . The coordinate axes are called respectively the real and imaginary axes. Addition of complex numbers coincides with the operation of vector sum (Figure 18).

The absolute value function has the geometrical meaning of the distance from the origin:  $|z| = \langle z, z \rangle^{1/2}$ . In particular, the triangle inequality  $|z + w| \leq |z| + |w|$  holds true. Complex numbers of unit absolute value  $|z| = 1$  form the unit circle centered at the origin.

The operation of complex conjugation acts on the radius-vectors  $z$  as the reflection about the real axis.

<sup>6</sup>This requires that a set be equipped with commutative and associative operations (called addition and multiplication) satisfying the **distributive law**  $z(v + w) = zv + zw$ , possessing the zero and unit elements 0 and 1, additive opposites  $-z$  for every  $z$ , and multiplicative inverses  $1/z$  for every  $z \neq 0$ .

In order to describe a geometric meaning of complex multiplication, let us study the way multiplication by a given complex number  $z$  acts on all complex numbers  $w$ , i.e. consider the function  $w \mapsto zw$ . For this, write the vector representing a non-zero complex number  $z$  in the **polar** (or trigonometric) form  $z = ru$  where  $r = |z|$  is a positive real number, and  $u = z/|z| = \cos \theta + i \sin \theta$  has absolute value 1 (see Figure 19). Here  $\theta = \mathbf{arg} z$ , called the **argument** of the complex number  $z$ , is the angle that  $z$  as a vector makes with the positive direction of the real axis.

Clearly, multiplication by  $r$  acts on all vectors  $w$  by stretching them  $r$  times. Multiplication by  $u$  applied to  $w = x + yi$  yields a new complex number  $uw = X + Yi$  according to the rule:

$$\begin{aligned} X &= \operatorname{Re} [(\cos \theta + i \sin \theta)(x + yi)] = x \cos \theta - y \sin \theta \\ Y &= \operatorname{Im} [(\cos \theta + i \sin \theta)(x + yi)] = x \sin \theta + y \cos \theta. \end{aligned}$$

Comparing with the formula (\*) in Section 2, we conclude that the transformation  $w \mapsto uw$  is the counter-clockwise rotation through the angle  $\theta$ .

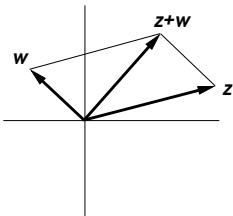


Figure 18

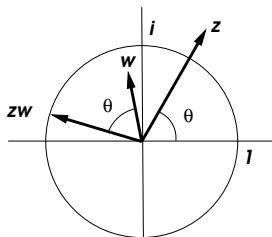


Figure 19

Notice a difference though: In Section 2, we rotated the coordinate system, and the formulas (\*) expressed old coordinates of a vector via new coordinates of the *same* vector. This time, we transform vectors, while the coordinate system remains unchanged. The same formulas now express coordinates  $(X, Y)$  of a new vector in terms of the coordinates  $(x, y)$  of a the old one.

Anyway, the conclusion is that multiplication by  $z$  is the composition of two operations: stretching  $|z|$  times, and rotating through the angle  $\mathbf{arg} z$ .

In other words, the product operation of two complex numbers sums their arguments and multiplies absolute values:

$$|zw| = |z| \cdot |w|, \quad \mathbf{arg} zw = \mathbf{arg} z + \mathbf{arg} w \text{ modulo } 2\pi.$$

For example, if  $z = r(\cos \theta + i \sin \theta)$ , then  $z^n = r^n(\cos n\theta + i \sin n\theta)$ .

## The Fundamental Theorem of Algebra

A degree 2 polynomial  $z^2 + pz + q$  has two roots

$$z_{\pm} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

This **quadratic formula** works regardless of the sign of the **discriminant**  $p^2 - 4q$ , provided that we allow the roots to be complex, and take in account **multiplicity**. Namely, if  $p^2 - 4q = 0$ ,  $z^2 + pz + q = (z + p/2)^2$  and therefore the single root  $z = -p/2$  has multiplicity two. If  $p^2 - 4q < 0$  the roots are complex conjugate with  $\operatorname{Re} z_{\pm} = -p/2$ ,  $\operatorname{Im} z_{\pm} = \pm\sqrt{|p^2 - 4q|}/2$ . The Fundamental Theorem of Algebra shows that not only justice has been restored, but that any degree  $n$  polynomial has  $n$  complex roots, possibly — multiple.

**Theorem.** *A degree  $n$  polynomial*

$$P(z) = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$$

*with complex coefficients  $a_1, \dots, a_n$  factors as*

$$P(z) = (z - z_1)^{m_1} \dots (z - z_r)^{m_r}.$$

*Here  $z_1, \dots, z_r$  are complex roots of  $P$ , and  $m_1, \dots, m_r$  their multiplicities,  $m_1 + \dots + m_r = n$ .*

A proof of this theorem deserves a separate chapter (if not a book). Many proofs are known, based on various ideas of Algebra, Analysis or Topology. We refer to [6] for an exposition of the classical proof due to Euler, Lagrange and de Foncenex, which is almost entirely algebraic. Here we merely illustrate the theorem with several examples.

**Examples.** (a) To solve the quadratic equation  $z^2 = w$ , equate the absolute value  $r$  and argument  $\theta$  of the given complex number  $w$  with those of  $z^2$ :

$$|z|^2 = \rho, \quad 2 \operatorname{arg} z = \phi + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots$$

We find:  $|z| = \sqrt{\rho}$ , and  $\operatorname{arg} z = \phi/2 + \pi k$ . Increasing  $\operatorname{arg} z$  by even multiples  $\pi$  does not change  $z$ , and by odd changes  $z$  to  $-z$ . Thus the equation has two solutions:

$$z = \pm \sqrt{\rho} \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right).$$

(b) The equation  $z^2 + pz + q = 0$  with coefficients  $p, q \in \mathbb{C}$  has two complex solutions given by the quadratic formula (see above), because according to Example (a), the **square root** of a complex number takes on two opposite values (distinct, unless both are equal to 0).

(c) The complex numbers  $1, i, -1, -i$  are the roots of the polynomial  $z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i)$ .

(d) There are  $n$  complex  **$n$ th roots of unity** (see Figure 20, where  $n = 5$ ). Namely, if  $z = r(\cos \theta + i \sin \theta)$  satisfies  $z^n = 1$ , then  $r^n = 1$  (and hence  $r = 1$ ), and  $n\theta = 2\pi k$ ,  $k = 0, \pm 1, \pm 2, \dots$ . Therefore  $\theta = 2\pi k/n$ , where only the remainder of  $k$  modulo  $n$  is relevant. Thus the  $n$  roots are:

$$z = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}, \quad k = 0, 1, 2, \dots, n - 1.$$

For instance, if  $n = 3$ , the roots are 1 and

$$\cos \frac{2\pi}{3} \pm i \sin \frac{2\pi}{3} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}.$$

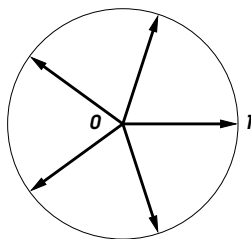


Figure 20

As illustrated by the previous two examples, even if all coefficients  $a_1, \dots, a_n$  of a polynomial  $P$  are real, its roots don't have to be real. But then the non-real roots come in pairs of complex conjugate ones. To verify this, we can use the fact that being real means stay *invariant* (i.e. unchanged) under complex conjugation. Namely,  $\bar{a}_i = a_i$  for all  $i$  means that

$$\overline{P(\bar{z})} = z^n + \bar{a}_1 z^{n-1} + \dots + \bar{a}_n = P(z).$$

Therefore we have two factorizations of the same polynomial:

$$\bar{P}(\bar{z}) = (z - \bar{z}_1)^{m_1} \dots (z - \bar{z}_r)^{m_r} = (z - z_1)^{m_1} \dots (z - z_r)^{m_r} = P(z).$$

They can differ only by orders of the factors. Thus, for each non-real root  $z_i$  of  $P$ , the complex conjugate  $\bar{z}_i$  must be also a root, and of the same multiplicity.

## Expanding the product

$$(z - z_1)\dots(z - z_n) = z^n - (z_1 + \dots + z_n)z^{n-1} + \dots + (-1)^n z_1 \dots z_n$$

we can express coefficients  $a_1, \dots, a_n$  of the polynomial in terms of the roots  $z_1, \dots, z_n$  (here multiple roots are repeated according to their multiplicities). In particular, the sum and product of the roots are

$$z_1 + \dots + z_n = -a_1, \quad z_1 \dots z_n = (-1)^n a_n.$$

These relations generalize **Vieta's theorem**  $z_+ + z_- = -p$ ,  $z_+ z_- = q$  about roots  $z_{\pm}$  of quadratic equations  $z^2 + pz + q = 0$ .

**EXERCISES**

**44.** Can complex numbers be: real? real *and* imaginary? neither? ✓

**45.** Compute: (a)  $(1 + i)/(3 - 2i)$ ; (b)  $(\cos \pi/3 + i \sin \pi/3)^{-1}$ . ✓

**46.** Verify the commutative and distributive laws for multiplication of complex numbers.

**47.** Show that  $z^{-1}$  is real proportional to  $\bar{z}$  and find the proportionality coefficient. ✓

**48.** Find all  $z$  satisfying the equations:  $|z - 1| = |z + 1| = 2$ . ✓

**49.** Sketch the solution set to the following system of inequalities:

$$|z - 1| \leq 1, \quad |z| \leq 1, \quad \operatorname{Re}(iz) \leq 0.$$

**50.** Compute absolute values and arguments of (a)  $1 - i$ , (b)  $1 - i\sqrt{3}$ . ✓

**51.** Compute  $\left(\frac{\sqrt{3}+i}{2}\right)^{100}$ . ✓

**52.** Express  $\cos 3\theta$  and  $\sin 3\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ . ♣

**53.** Express  $\cos(\theta_1 + \theta_2)$  and  $\sin(\theta_1 + \theta_2)$  in terms of  $\cos \theta_i$  and  $\sin \theta_i$ .

**54.** Prove **Bézout's theorem**<sup>7</sup>: A number  $z_0$  is a root of a polynomial  $P$  in one variable  $z$  if and only if  $P$  is divisible by  $z - z_0$ . ♣

**55.** Find roots of degree 2 polynomials:

$$z^2 - 4z + 5, \quad z^2 - iz + 1, \quad z^2 - 2(1 + i)z + 2i, \quad z^2 - 2z + i\sqrt{3}. \quad \checkmark$$

**56.** Find all roots of polynomials:

$$z^3 + 8, \quad z^3 + i, \quad z^4 + 4z^2 + 4, \quad z^4 - 2z^2 + 4, \quad z^6 + 1. \quad \checkmark$$

**57.** Prove that every polynomial with real coefficients factors into the product of polynomials of degree 1 and 2 with real coefficients. ♣

**58.** Prove that the sum of all 5th roots of unity is equal to 0. ♣

**59.\*** Find general **Vieta's formulas**<sup>8</sup> expressing all coefficients of a polynomial in terms of its roots. ✓

<sup>7</sup>Named after Étienne **Bézout** (1730–1783).

<sup>8</sup>Named after François Viète (1540–1603) also known as Franciscus **Vieta**.