Complex numbers

Kronecker: "God made the integers, all else is the work of man."

$\mathbb{N}$ - natural numbers $= \{0, 1, 2, 3, \dots\}$?

Greeks/Arnold vs. Bourbaki
(Archimedes, Kabbalah, ...)

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ - to make subtraction always possible.

$\mathbb{Q} = \{\pm \frac{m}{n}\}$ - to make division by non-zero numbers always possible.

$\mathbb{R}$ = reals - to "fill-in the gaps" in $\mathbb{Q}$.

$\mathbb{Q}$ - ordering, distance $|\alpha - \beta|$, decimals $3.14 = \frac{314}{100}$

$\Downarrow$ Dedekind's cuts, $\Downarrow$ Completion, $3.1415926\dots$

Miracle: all the 3 ways give the same result.

$\mathbb{C}$ = complex numbers

al-Khwarizmi:  ?? given area A perimeter P

$x^2 - \frac{P}{2}x + A = 0$, $x = ?, ??$

$x^2 + 1 = 0 \Rightarrow x = \pm i$

$\mathbb{C} = \{ a + bi \ \cancel{(+ci^2 + di^3 + \dots)} \}$

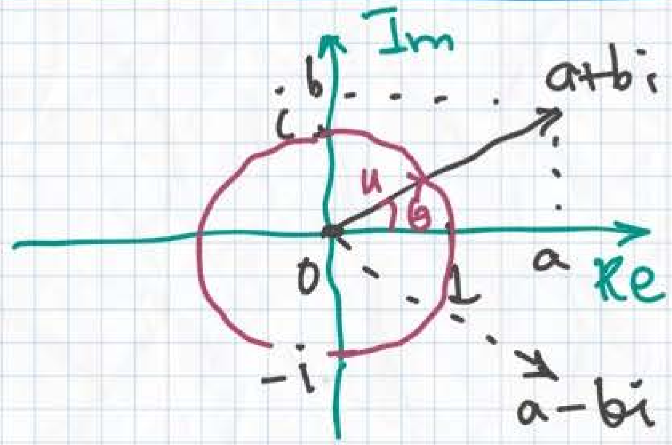
$$\begin{array}{r} a+bi \\ c+di \\ \hline (a+c)+(b+d)i \end{array} \qquad (a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

$i \mapsto -i$ · an automorphism of $\mathbb{C} \supset \mathbb{R}$ fixed points

$(a+bi)(a-bi) \in \mathbb{R}_{>0}$ unless $a=b=0$ (is it a miracle?)

$$\Rightarrow \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Geometric interpretation



Multiplication by $a+bi$

$$\mathbb{C} \longrightarrow \mathbb{C}$$

$$x+iy \mapsto (ax-by) + (ay+bx)\,i$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$a+bi = \sqrt{a^2+b^2}\; u = \sqrt{a^2+b^2}\,(\cos\theta + i\sin\theta)$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

counter-clockwise rotation through θ

stretch $|a+bi|$ times

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos\theta + i\sin\theta = \sum_{n\geq 0} \frac{(i\theta)^n}{n!} = e^{i\theta}$$

$$a+bi =: z = \underbrace{|z|}_{\sqrt{a^2+b^2}}\, e^{i\arg z}, \quad \arg z = \arctan b/a$$

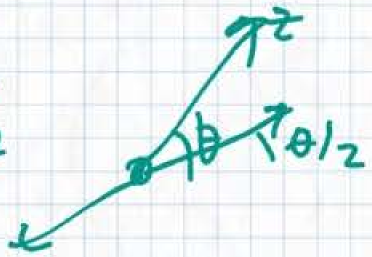
$$|zw| = |z|\,|w|$$

$$\arg(zw) \equiv \arg z + \arg w \mod 2\pi$$

Example 1.

$$z = |z|\, e^{i\theta}$$

$$\sqrt{z} = \sqrt{|z|}\; e^{i(\theta+2\pi k)/2} = \pm\sqrt{|z|}\, e^{i\theta/2}$$



Example 2.

$$x^2 + px + q = 0 \Rightarrow \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4} = 0$$

$$\Rightarrow x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

– makes sense for all $p, q \in \mathbb{C}$

Miracle: The Fundamental Theorem of Algebra

Not only all quadratics, but all polynomial equations with coeff. in $\mathbb{C}$ have all their roots in $\mathbb{C}$. Spoiler: To be proved in this course.

Formal Power Series

$$A(X) := \sum_{n\geq 0} a_n X^n = a_0 + a_1 X + a_2 X^2 + \dots, \quad a_n \in \mathbb{C}$$

$$B(X) := \sum b_n X^n$$

$$A(X) + B(X) := \sum_{n\geq 0} (a_n + b_n) X^n$$

Order $\omega(A)$: (undefined or $=\infty$ if $A=0$)

$$A(X) = \underbrace{\cancel{a_0}}_{0} + \underbrace{\cancel{a_1}}_{0} X + \dots + \underbrace{\cancel{a_{\omega-1}}}_{0} X^{\omega-1} + \underbrace{a_\omega}_{\neq 0} X^\omega + \dots$$

Infinite sums of summable families:

$$S_i(X) = \sum_{n\geq 0} a_{n,i} X^n, \quad i \in I$$

$$\sum_i S_i(X) = S(X), \quad a_n = \sum_i a_{n,i}$$

provided that for each n all but finitely many $a_{n,i} = 0$

≡ Equivalently, for every k, $\omega(S_i) \geq k$ for all but finitely many indices i.

Multiplication: $C(X) = A(X)\,B(X)$

$$a_k X^k \times b_\ell X^\ell = a_k b_\ell X^{k+\ell} \text{ - summable family}$$

$$C(X) = \sum_{k,\ell} a_k b_\ell X^{k+\ell} = \sum_{n\geq 0} \underbrace{\left(\sum_{k+\ell=n} a_k b_\ell\right)}_{c_n} X^n$$

Proposition: $\omega(AB) = \omega(A) + \omega(B)$

$$(\underbrace{a_\omega}_{\neq 0} X^\omega + \dots)(\underbrace{b_{\omega'}}_{\neq 0} X^{\omega'} + \dots) = \underbrace{a_\omega b_{\omega'}}_{\Rightarrow\ \neq 0} X^{\omega+\omega'} + \dots$$

higher order terms

Corollary $\mathbb{C}[[X]]$ - an integral domain

= commutative ring ($\mathbb{C}$-algebra) with unity 1 without zero divisors
$(1 + 0X + 0X^2 + \dots)$

Composition of Power Series

$$S(X) = \sum_{n\geq 0} a_n X^n \qquad X = T(Y) = \sum_{p\geq 1} b_p Y^p \qquad b_0 = 0$$

$a_n T(Y)^n = a_n b_1^n Y^n + \dots$ - summable family

$$(S\circ T)(Y) := \sum_{n\geq 0} a_n T(Y)^n$$

$\mathbb{C}[[X]] \xrightarrow[\smile\circ T]{} \mathbb{C}[[Y]]$ - ring homomorphism

$(S_1+S_2)\circ T = S_1\circ T + S_2\circ T$, $(S_1S_2)\circ T = (S_1\circ T)(S_2\circ T)$

Check for monomials S_1, S_2.

Moreover: $\left(\sum S_i\right)\circ T = \sum (S_i\circ T)$

if the family is summable.

$$\sum_{n\geq 0}\left(\sum_i a_{n,i}\right) T(Y)^n \overset{?}{=} \sum_i \left(\sum_{n\geq 0} a_{n,i} T(Y)^n\right)$$

Comparing coeff. at Y^p involves finitely many S_i

Corollary. Composition is associative:

$(S\circ T)\circ U = S\circ (T\circ U)$ provided that $\omega(T), \omega(U) > 0$

$$\begin{tikzcd} \mathbb{C}[[X]] \arrow[r, "\smile\circ T"'] \arrow[rr, bend right, "\smile(T\circ U)"'] & \mathbb{C}[[Y]] \arrow[r, "\smile\circ U"'] & \mathbb{C}[[Z]] \end{tikzcd}$$

$$X \mapsto T(Y) \mapsto (T\circ U)(Z)$$

$\Rightarrow$ true for monomials $\Rightarrow$ true for $\sum a_n X^n$

Algebraic inverses: Exist iff $\omega(S) = 0$

$$\frac{1}{S(X)} = \frac{1}{a_0}\frac{1}{1-U(X)} = \frac{1}{a_0}\left(1+U(X)+U(X)^2+U(X)^3+\dots\right)$$

$\omega(U)\geq 1.$

Formal derivative: $\frac{d}{dX}S = \sum_{n\geq 0} n a_n X^{n-1}$ - linear

$\frac{d}{dX}(ST) = \frac{dS}{dX}T + S\frac{dT}{dX}$ suffices to check for monomials

$\frac{d}{dX}\left(\frac{1}{S}\right) = -\frac{1}{S^2}\frac{dS}{dX}$ if $\omega(S)=0$

$\Leftarrow \frac{d}{dX} S\left(\frac{1}{S}\right) = 0$

The Formal Inverse Function Theorem

Given a formal series S, a necessary and sufficient condition for there to exist a formal series T such that $T(0)=0$ and $S(T(Y))=Y$ is that $S(0)=0$ and $\boxed{S'(0)\neq 0}$. In this case, T is unique and $T(S(X))=X$. (i.e. T is inverse to S)

Proof. $(\Rightarrow)$ $a_0 + a_1(b_1 Y + \dots) + \dots = Y$

$\Rightarrow a_0 = 0, \quad a_1 b_1 = 1.$

$(\Leftarrow)$ $\underset{0\neq}{a_1} X + a_2 X^2 + \dots = Y \rightsquigarrow X = T(Y)$?

$$(*) \quad X_{n+1} = \frac{Y}{a_1} - \frac{a_2}{a_1} X_n^2 - \frac{a_3}{a_1} X_n^3 - \dots$$

iterations!

$n=0$ $\quad X_0 = 0 \mod Y^1 \Rightarrow X_1 = \frac{Y}{a_1} \mod Y^2$

$n=1$ $\quad \Rightarrow X_2 = \frac{Y}{a_1} - \frac{a_2}{a_1}\left(\frac{Y}{a_1} + \dots\right)^2 + \dots \mod Y^3$

$\vdots$

Lemma If X_n is a fixed point of $(*)$ mod Y^{n+1}, then X_{n+1} is a fixed point of $(*)$ mod Y^{n+2}

$$X_n + \alpha Y^{n+1} + \dots \equiv Y/a_1 - \frac{a_2}{a_1}\Big(X_n^2 - \frac{a_3}{a_1} X_n^3 \quad \mod Y^{n+2}$$

mod Y^{n+1}

Corollary: $X_{n+1} \equiv X_n \mod Y^{n+1}$

Corollary: $X_\infty = T(Y)$ exists and is unique (since $X_\infty \equiv X_n \mod Y^{n+1}$ for all n).

Finally: Find S_1 s.t. $T(S_1(X)) = X$

Then $S_1 = I \circ S_1 = (S \circ T) \circ S_1$
$= S \circ (T \circ S_1) = S \circ I = S$

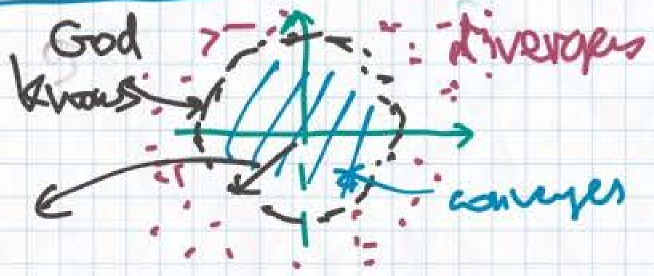
$\Rightarrow T \circ S = I$

Convergence disk of a power series

3.2

$\sum_{n\geq 0} a_n z^n$, $a_n \in \mathbb{C}$

$1/\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \rho$



① $\underbrace{c_0 + c_1 + \dots + c_n}_{S_n} + c_{n+1} + \dots + c_m + \dots$ - converges $\iff \lim S_n$ exists.

$\Rightarrow c_n = S_n - S_{n-1} \to 0$ $\quad \not\Leftarrow \; 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots = \infty$

② $|c_n| \leq a_n$, $\sum a_n < \infty \Rightarrow \sum c_n$ converges

$|S_m - S_n| \leq |c_{n+1}| + \dots + |c_m| \leq a_{n+1} + \dots + a_m$

Cauchy: $\forall \varepsilon > 0 \; \exists N$ s.t. $\forall m > n \geq N$ $|S_m - S_n| < \varepsilon$

②a $\sum |c_n| < \infty \Rightarrow \sum c_n$ - converges. $\quad \not\Leftarrow \; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

(abs. conv. $\Rightarrow$ conv.)

②b Suppose $\sqrt[n]{|c_n|} \leq \beta < 1$ for all $n \geq n_0$

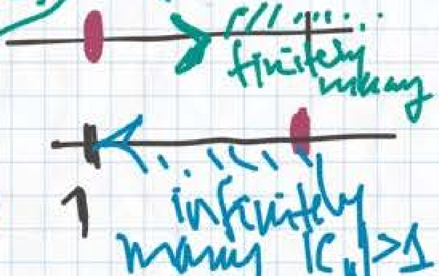
Then $\sum c_n$ conv. absolutely $\quad |c_n| \leq \beta^n$

③ $\limsup a_n$ $\quad + \delta > 0$ $\quad$ finitely many a_n

$-\delta$ $\quad$ infinitely many a_n

④ The Root Test:

$\limsup_{n\to\infty} \sqrt[n]{|c_n|} \begin{cases} < 1 & \text{converges}: \; \beta < 1 \\ > 1 & \text{diverges} \end{cases}$



⑤ $\sum_{n \geq 0} c_n z^n$

$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} =: \frac{|z|}{\rho} \lessgtr 1$

⑥ For $(0 \leq) \beta < \rho$, $\sum c_n z^n$ converges normally ($\Rightarrow$ absolutely & uniformly) on $|z| \leq \beta$

$\sum_{n\geq 0} v_n(z)$, $\|v_n\|_E := \sup_{z \in E} |v_n(z)|$, $\sum_{n \geq 0} \|v_n\|_E < \infty$

$\Rightarrow \sum c_n z^n$ is continuous on $|z| \leq \beta < \rho$.

Operating with convergent power series

$\sum_{n\geq 0} c_n z^n$ converges for $|z|<$, diverges for $|z|>$ $\rho = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$

The same for $\sum_{n\geq 0} |c_n| z^n$ and $z = r > 0$

$\sum_{n\geq 0} |c_n| r^n < \infty$ for $r<\rho$, $= \infty$ for $r>\rho$

$A = \sum a_n z^n$, $B = b_n z^n$, $S = A+B$

$\sum |a_n+b_n| r^n \leq \sum |a_n| r^n + \sum |b_n| r^n$

$\Rightarrow \rho_S \geq \min(\rho_A, \rho_B)$

$P = A \cdot B = \sum c_n z^n$, $c_n = \sum_{k+l=n} a_k b_l$

$\sum |c_n| r^n \leq \sum \left(\sum_{k+l=n} |a_k||b_l|\right) r^n = \left(\sum |a_k| r^k\right)\left(\sum |b_l| r^l\right)$

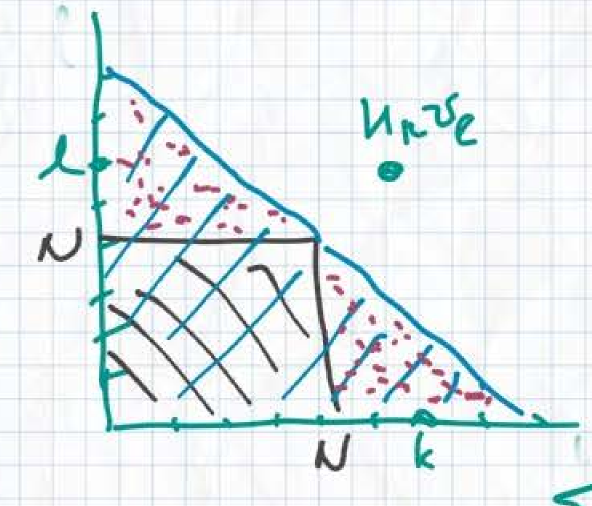
$\Rightarrow \rho_P \geq \min(\rho_A, \rho_B)$

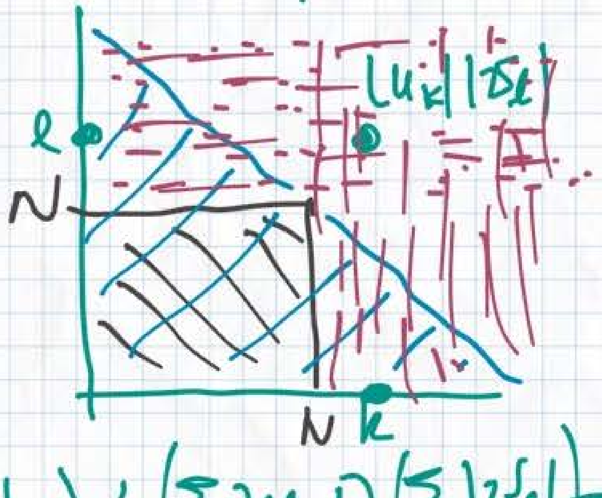
Obviously $S(z) = A(z) + B(z)$ for $|z| < \rho_A, \rho_B$

Claim: $P(z) = A(z)\,B(z)$ for $|z| < \rho_A, \rho_B$

Lemma $\sum |u_k| = U < \infty$, $\sum |v_l| = V < \infty$, $w_n = \sum_{k+l=n} u_k v_l$

$\Rightarrow \sum |w_n| < \infty$ and $\sum w_n = \left(\sum u_k\right)\left(\sum v_l\right)$





$\sum |w_n| \leq \left(\sum |u_k|\right)\left(\sum |v_l|\right) = UV$

$$\left| \sum_0^{2N} w_n - \left(\sum_0^N u_k\right)\left(\sum_0^N v_l\right) \right| \leq \left(\sum_{k>N} |u_k|\right)\left(\sum |v_l|\right) + \left(\sum |u_k|\right)\left(\sum_{l>N} |v_l|\right)$$

$\xrightarrow[N\to\infty]{} 0 \cdot V + U \cdot 0$

Differentiation $\sum n a_n z^{n-1}$ has the same convergence radius as $\sum a_n z^n$

$r \cdot \sum n |a_n| r^{n-1} < \infty \Leftrightarrow r < 1/\underbrace{\lim \sqrt[n]{n}}_{=1} \cdot \limsup \sqrt[n]{|a_n|}$

Moreover $S'(z) = \lim\limits_{h \to 0} \dfrac{S(z+h) - S(z)}{h}$

$\dfrac{S(z+h)-S(z)}{h} - S'(z)$ $\qquad a^n - b^n = (a-b)(a^{n-1} + a^{n-2} b + \dots b^{n-1})$

$= \sum\limits_{n>0} a_n \Big[\underbrace{(z+h)^{n-1} + (z+h)^{n-2} z + \dots + z^{n-1} - n z^{n-1}}_{\text{vanishes at } h=0}\Big]_n$

$|[\dots]_n| < 2n r^{n-1}$ if $|z|, |z+h| \le r < \rho$

$\Rightarrow \left|\sum\limits_{n>0} a_n [\ ]_n\right| \le \underbrace{\left|\sum\limits_{n=1}^{N} a_n [\ ]_n\right|}_{<\varepsilon/2 \text{ for small } h} + \underbrace{2 \sum\limits_{n>N} |a_n| n r^{n-1}}_{<\varepsilon/2 \text{ for large } N}$

Composition

$T(z) = \sum\limits_{n \ge 1} b_n z^n \qquad S(T(z)) = \sum a_p T(z)^p$

$r \sum |b_n| r^{n-1} < \rho_S \Rightarrow \sum |a_p| \left(\sum |b_n| r^n\right)^p < \infty$

for small $r < \rho_T$ $\qquad \Rightarrow r \le \rho_{S \circ T}$

Moreover $U_n(z) := S_n(T(z)) \longrightarrow S(T(z))$ for $|z| \le r$

S_n partial sum

$|U(z) - U_n(z)| \le \sum\limits_{p>n} |a_p| \left(\sum |b_k| r^k\right)^p \xrightarrow[n\to\infty]{} 0$

$\Rightarrow U(z) = \lim U_n(z) = S(T(z))$ for $|z| \le r$.

Corollary (Division) $\rho_S > 0, S(0) \ne 0 \Rightarrow \rho_{1/S} > 0$.

$\dfrac{1}{1-U(z)} = 1 + U(z) + U^2(z) + \dots, \quad U(0) = 0, \ \rho_U > 0$

Theorem $w = S(z), \ \rho_S > 0 \Rightarrow z = T(w), \ \rho_T > 0$

$x_{n+1} = w - a_2 x_n^2 - a_3 x_n^3 - \dots \qquad (\rho_S > 1) \Downarrow$

$\bar{x}_{n+1} = w + M \bar{x}_n^2 + M \bar{x}_n^3 + \dots \qquad |a_n| \le M$

$\bar{x} = \sum B_n w^n, \ |b_n| \le B_n \quad \bar{x} = \dfrac{w+1 - \sqrt{(w+1)^2 - 4(M+1)w}}{2(M+1)}$

The Analytical Inverse Function Theorem

$$w = S(z) = a_1 z + a_2 z^2 + \dots, \quad \rho_S > 0$$

$$\Rightarrow \rho_{S^{-1}} > 0, \quad z = S^{-1}(w) = b_1 w + b_2 w^2 + \dots$$

① $r \le \rho_S \rightleftarrows \exists M : |a_n| \le \frac{M}{r^n}$ for all n

$\limsup_{n\to\infty} \sqrt[n]{|a_n|}\, r < 1 \Rightarrow |a_n| < \frac{1}{r^n}$ for $n \ge n_0$

② Rescaling z: WLOG, $\rho_S > 1 \Rightarrow |a_n| \le M$

③ Rescaling w: WLOG, $a_1 = 1$.

$z_{n+1} = w - a_2 z_n^2 - a_3 z_n^3 + \dots$	$z_{n+1} = w + A_2 z_n^2 + A_3 z_n^3 + \dots$
$z = w + b_2 w^2 + b_3 w^3 + \dots$	$z = w + B_2 w^2 + B_3 w^3 + \dots$

If for all k, $|a_k| \le A_k$, then $|b_k| \le B_k$.

Take all $A_k = M$: $z = w + Mz^2/1-z$

$$(M+1)z^2 - (w+1)z + w = 0$$

$$z = \left[w+1 \pm \sqrt{(w+1)^2 - 4(M+1)w}\right]/2(M+1)$$

$$(1+\alpha)^{1/2} = 1 + \frac{\alpha}{2} - \frac{\alpha^2}{8} + \dots \frac{(\frac{1}{2})(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}\alpha^n + \dots$$ Converges for $|\alpha| < 1$

Exponential Function: $e^z = \sum_{n\ge 0} z^n/n!$

The Ratio Test: If $\limsup_{n\to\infty} |c_{n+1}|/|c_n| < 1$ then $\sum c_n$ converges absolutely.

Proof: $|c_{n+1}| \le \beta |c_n| \;\forall n \ge n_0, \; \beta < 1$

$$\Rightarrow |c_{n+n_0}| \le |c_{n_0}|\beta^n \Rightarrow \sum_{n\ge n_0} |c_n| \le \frac{|c_{n_0}|}{1-\beta}$$

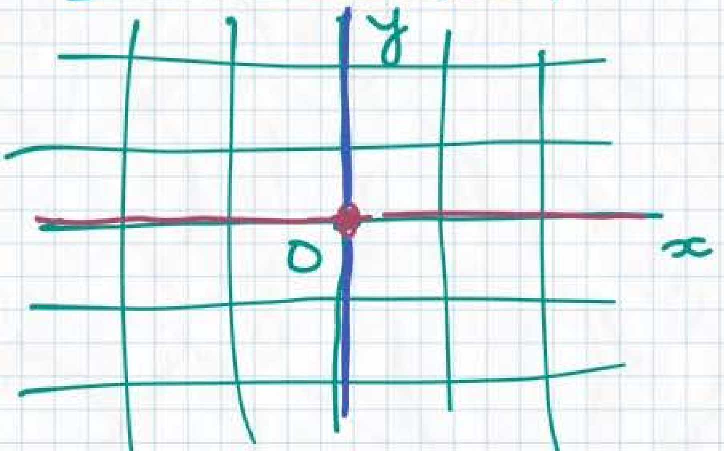
$$\frac{|z^{n+1}|}{(n+1)!} \Big/ \frac{|z^n|}{n!} = \frac{|z|}{n+1} \to 0 \Rightarrow \rho = \infty.$$

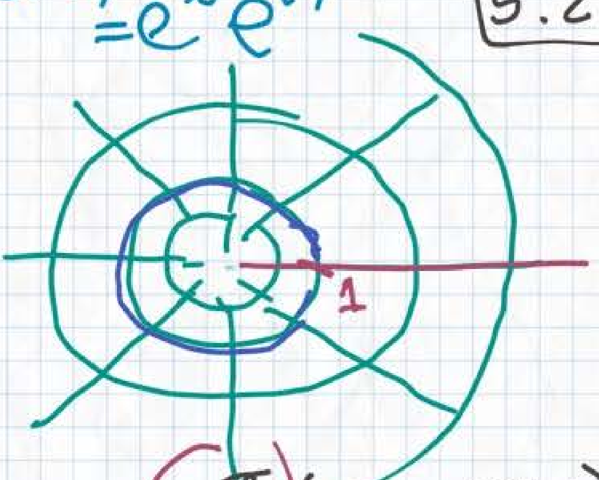
exp: $\mathbb{C} \to \mathbb{C}^\times$ - group homomorphism

$$e^z e^w = \sum_{n\ge 0}\left(\sum_{k+l=n} \frac{z^k w^l}{k!\,l!}\right) = \sum_{n\ge 0} \frac{1}{n!}(z+w)^n = e^{z+w}$$

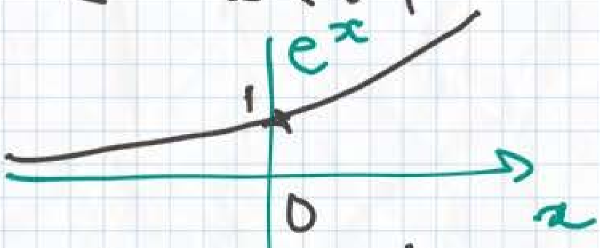
$$(z+w)\dots(z+w) = \dots + \binom{n}{k} z^k w^{n-k} \dots \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The geometry of $e^{x+iy} = e^x e^{iy}$

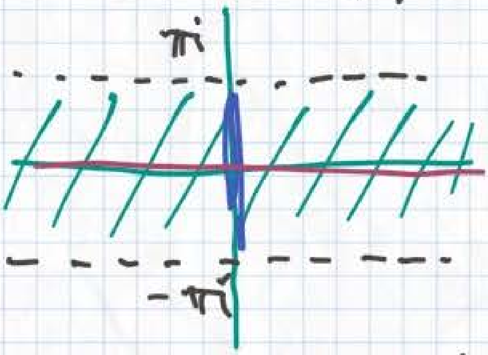


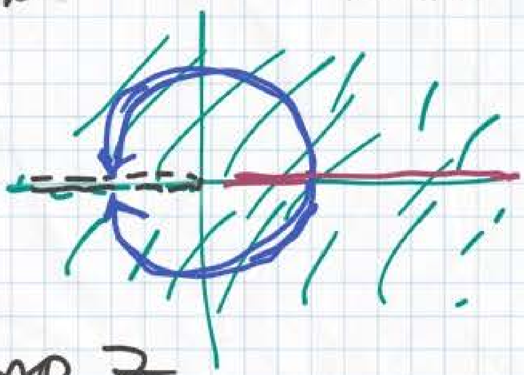


$z = x + iy$ $\qquad$ $w = e^x(\cos y + i \sin y)$



$$e^{iy} = \sum \frac{(iy)^n}{n!} = \overbrace{\sum_{n=2k} \frac{(-1)^k y^{2k}}{2k!}}^{\cos y} + i \overbrace{\sum_{n=2k+1} (-1)^k \frac{y^{2k+1}}{(2k+1)!}}^{\sin y}.$$

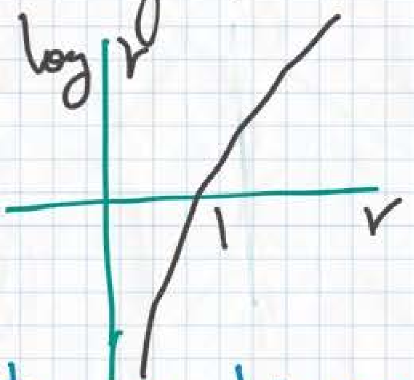




$$\exp(z + 2\pi i) = \exp z$$

The Complex logarithm: $z = \log w$

$$\log |w| e^{i \arg w} \equiv \log |w| + i \arg w \mod 2\pi i$$



$$\log w_1 w_2 \equiv \log w_1 + \log w_2 \mod 2\pi i$$

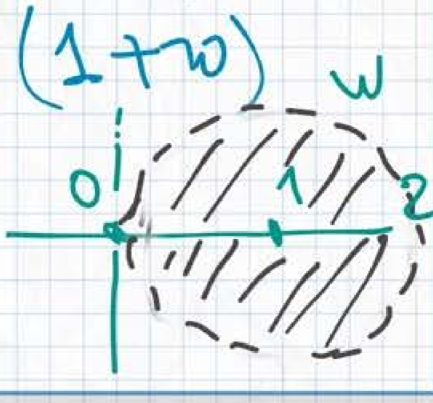
Derivatives: $\frac{d}{dz} e^z = e^z$, $\frac{d}{dw} \log w = \frac{1}{w}$

$$\frac{d}{dz} \sum a_n T(z)^n = \sum n a_n T^{n-1}(z) \frac{dT}{dz} = S'(T(z))\, T'(z)$$

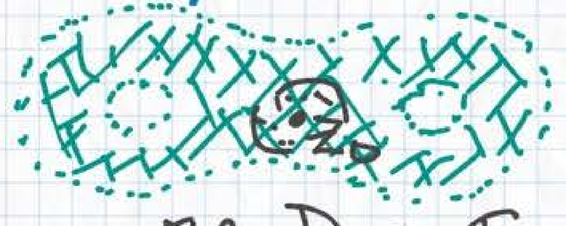
Series expansion of $\log(1+w)$

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \dots$$

$$\log(1+w) = 0 + w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \dots$$



Analytic functions



open $D \subset \mathbb{C}$ $\xrightarrow{f} \mathbb{C}$ is called analytic

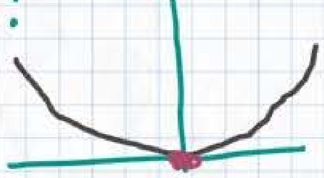
if $\forall z_0 \in D$, $\exists \sum_{n \geq 0} a_n (z-z_0)^n = f\big|_{|z-z_0|<\rho}$

Analyticity $\Rightarrow$ (infinite) differentiability

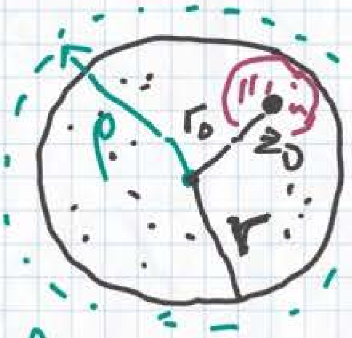
$\Updownarrow$

f = its Taylor series $\quad \not\Leftarrow \quad a_n = f^{(n)}(z_0)/n!$

$f(x) = e^{-1/|x|}$

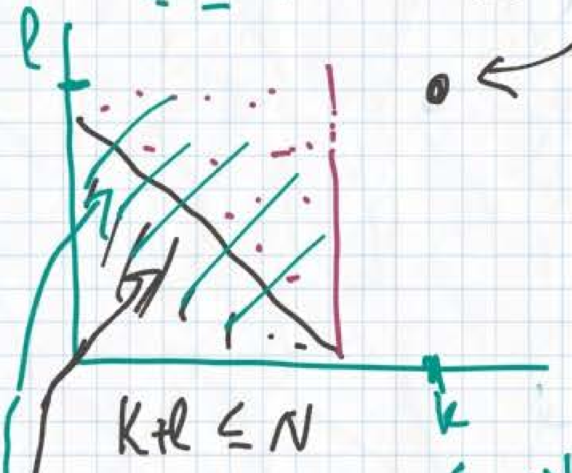


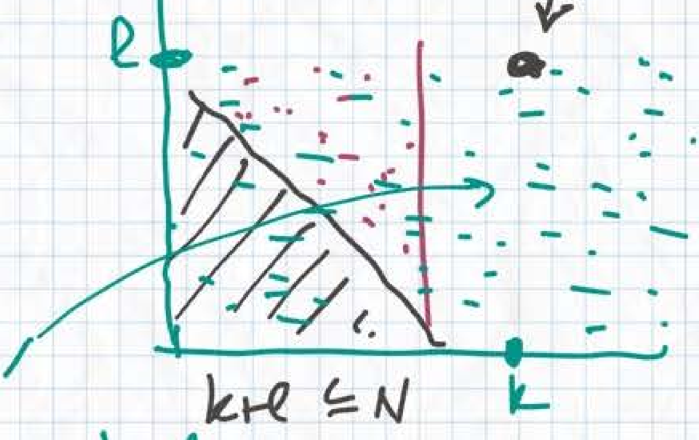
Theorem: $S(z) := \sum_{n \geq 0} a_n z^n$, $\rho > 0$, is analytic for $|z_0| < \rho$.



$|z_0| = r_0 < r < \rho$

$$\Big| \underbrace{a_{k+\ell} \frac{(k+\ell)!}{k!\,\ell!} (z-z_0)^k z_0^\ell}_{} \Big| \leq \underbrace{|a_{k+\ell}| \frac{(k+\ell)!}{k!\,\ell!} (r-r_0)^k r_0^\ell}_{}$$



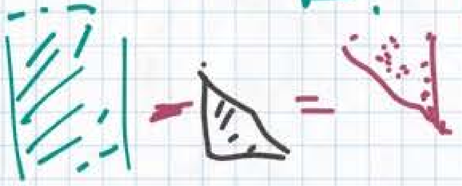


$$\sum_{n \geq 0} |a_n| \sum_{k+\ell=n} \frac{(k+\ell)!}{k!\,\ell!} (r-r_0)^k r_0^\ell = \sum_{n \geq 0} |a_n| r^n < \infty$$

$$\sum_{n \leq N} a_n \sum_{k+\ell=n} \frac{(k+\ell)!}{k!\,\ell!} (z-z_0)^k z_0^\ell = \sum_{n \leq N} a_n z^n \xrightarrow[N\to\infty]{} S(z)$$

$$\sum_{k \leq N} \frac{(z-z_0)^k}{k!} \Big(\sum_{\ell \geq 0} a_{k+\ell} \frac{(k+\ell)!}{\ell!} z_0^\ell \Big) = S^{(k)}(z_0)$$

$$= \sum_{k \leq N} S^{(k)}(z_0) \frac{(z-z_0)^k}{k!} \xrightarrow[N\to\infty]{} \sum_{k \geq 0} \frac{S^{(k)}(z_0)}{k!} (z-z_0)^k$$



$\to 0$ as $N \to \infty$

Analytical continuation

An analytic function on a connected open D is uniquely determined by Taylor coeff. at a pt.

$f^{(n)}(z_0)=0 \ \forall n\geq 0 \Rightarrow f(z)\equiv 0$ for $|z-z_0|<\rho$

$\Rightarrow \{z\in D \mid f^{(n)}(z)=0 \ \forall n\geq 0\}$ - open - closed $= D$

Corollary: $\underbrace{\mathcal{A}(D)}_{\text{analytic functions } D\to\mathbb{C}}$ - an integral domain

(b/c true for formal series at any $z_0\in D$).

Zeroes of non-zero analytic funct. are isolated

$$f(z) = \underbrace{(z-z_0)}_{\substack{\neq 0 \\ z\neq z_0}}{}^{k}\underbrace{[\underset{\neq 0}{a}+b(z-z_0)+c(z-z_0)^2+\dots]}_{\neq 0 \text{ in some disk}}$$

k - order of zero

Corollary: An analytic function on a connected open D is uniquely determined by its values on a sequence of pts converging in D.

$z_n\to z_0\in D$, $f(z_n)=0 \Rightarrow f\equiv 0$.

The field $\mathcal{M}(D)$ of meromorphic functions

$$\frac{f}{g}=\frac{(z-z_0)^k(\overset{\neq 0}{\alpha}+\beta(z-z_0)+\dots)}{(z-z_0)^\ell(\underset{\neq 0}{\gamma}+\delta(z-z_0)+\dots)}=(z-z_0)^{k-\ell}\left(\frac{\alpha}{\gamma}+\dots\right)$$

order of pole if $k<\ell$.

Def: Meromorphic in D: analytic except (isolated) poles.

If D is connected, then $\mathcal{M}(D)$ is a field

$$\frac{d}{dz}:\mathcal{M}(D)\to\mathcal{M}(D)$$

$$\frac{d}{dz}\frac{g(z)}{(z-z_0)^\ell}=\underbrace{\frac{g'(z)}{(z-z_0)^\ell}-\frac{\ell g(z)}{(z-z_0)^{\ell+1}}}_{\text{pole of order } \ell+1}$$

$g(z_0)\neq 0$

Examples: Rational $\frac{P(z)}{Q(z)}$, e^z, $\frac{e^{iz}+e^{-iz}}{2}=\cos z$

$\tan z=\sin z/\cos z$, branches of $\log z$ [Σ is not]

Holomorphic functions $f: D \to \mathbb{C}$

Def. f is called holomorphic at $z_0 \in D \subset \mathbb{C}$ open
if $f'(z_0) := \lim\limits_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists.

Example: $f(z) = \sum a_n z^n$, $\rho > 0$; $f'(z) = \sum n a_n z^{n-1}$
– in fact ∞-differentiable in $|z| < \rho$.

"Real" point of view: $\mathbb{R}^2 \supset D \xrightarrow{f} \mathbb{C} = \mathbb{R}^2$

$f = u(x,y) + i v(x,y)$, $z = x + iy$
is called differentiable at (x_0, y_0) if $\exists \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$:

$$|u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - \alpha\Delta x - \beta \Delta y| / \sqrt{\Delta x^2 + \Delta y^2} \to 0 \text{ as } \sqrt{\cdots} \to 0$$

$$|v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0) - \gamma\Delta x - \delta \Delta y| / \sqrt{\Delta x^2 + \Delta y^2} \to 0$$

$$|f(z_0+h) - f(z_0) - \underbrace{(a+bi)}_{f'(z_0)} \underbrace{h}_{\Delta x + i\Delta y}| / |h| \to 0 \text{ as } |h| \to 0.$$

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \mapsto \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad h \mapsto (a+bi)h \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Cauchy-Riemann eqns:

$f = u + iv$ is holomorphic (at z_0) iff
(u,v) is differentiable and $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$

Example. Suppose u, v are polynomial in x, y.
(or, $f(x,y)$ – complex coeff. polyn. in x, y)
Which polynomials are holomorphic?
$z = x + iy$, $\bar{z} = x - iy$, $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$

$$f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = g(z, \bar{z}) = \sum g_{k\ell} z^k \bar{z}^\ell$$

Claim: f is holomorphic iff
g does not depend on $\bar{z}$: $g_{k,\ell} = 0$ for $\ell > 0$.

$\bar{z} = x - iy$ $\quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Holomorphy $\iff$ "independence of $\bar z$"

$x = \frac{z+\bar z}{2}$ $\qquad \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]$

$y = \frac{z-\bar z}{2i}$ $\qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$

Cauchy-Riemann eqns. $\boxed{\partial f/\partial \bar z = 0}$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \underbrace{(u_x - v_y)}_{=0} + i\,\underbrace{(u_y + v_x)}_{=0} = 0$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) = \underbrace{\frac{u_x+v_y}{2} + i\,\frac{v_x-u_y}{2}}_{a+bi} = \frac{df}{dz}$$

Preview: Holomorphic functions are analytic

- differentiability $\Rightarrow$ ∞-differentiability !!!
- "elliptic regularity" of C-R equations
- f-holom. $\Rightarrow \exists\, g$ s.t. $g' = f$ locally – within $|z|<\rho$

Global counter-example: $f = \frac{1}{z}$, $g = \log z$

Finding g, primitive of $f(z)dz$

$$f\,dz = (u+iv)(dx+idy) = \underbrace{(u\,dx - v\,dy)}_{dA} + i\,\underbrace{(v\,dx + u\,dy)}_{dB}$$

Is there $g = A + iB$ s.t. dA dB ?

Suppose that u, v are continuously differentiable

Then the necessary & locally sufficient condition is given by the "Clairaut's Test"

$$u_y = (-v)_x \quad \text{and} \quad v_y = u_x$$

Moreover: $dg = g_x\,dx + g_y\,dy = f\,dz$

$$= \left(\frac{1}{2}g_x + \frac{1}{2i}g_y\right)dz + \left(\frac{1}{2}g_x - \frac{1}{2i}g_y\right) = g_z\,dz + \underbrace{g_{\bar z}\,d\bar z}_{=0}$$

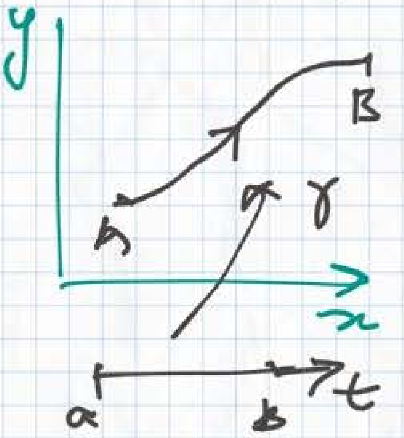
The primitive g is holomorphic.

"Cauchy's Thm": The continuity hypothesis is redundant!

Finding primitives: $\underbrace{P\,dx + Q\,dy}_{\text{differential 1-form}} = dF$ 18.1

$F(A) = \int_{A_0}^{A} P\,dx + Q\,dy$

Line integrals: γ – continuous piecewise continuously differentiable parametric curve



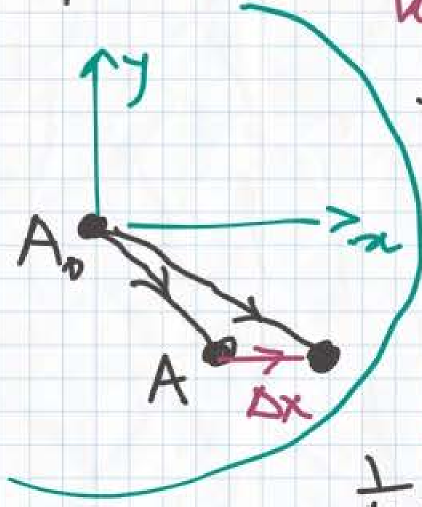
$$\int_A^B P\,dx + Q\,dy := \int_a^b [P(x(t),y(t))\,\dot{x}(t) + Q(x(t),y(t))\,\dot{y}(t)]\,dt$$

- does not change under re-parametrization $t = \varphi(\tau)$, with $\varphi' \geq 0$, $\int_a^b [\ ]\,dt = \int_\alpha^\beta [\ ]\,\varphi'(\tau)\,d\tau$
- changes the sign if $\varphi' \leq 0$ (orientation of γ)

Primitives: P, Q – continuously differentiable,

$P_y = Q_x \Rightarrow \exists\, F$ s.t. $F_x = P$, $F_y = Q$

locally!



$$F(A) :\overset{?}{=} \int_{A_0}^{A} P\,dx + Q\,dy$$

$$\frac{F(x+\Delta x, y) - F(x,y)}{\Delta x} \overset{?}{=} \frac{1}{\Delta x}\int_0^{\Delta x} P(x+t, y)\,dt \xrightarrow[\Delta x \to 0]{} P(x,y)$$

provided that $\int P\,dx + Q\,dy = 0$

$\overset{\text{Green}}{=} \iint (Q_x - P_y)\,dx\,dy$ if P, Q are continuously differentiable.

$= 0$ if $Q_x = P_y$

Moral: It suffices to assume that P, Q are merely continuous, but

$\int P\,dx + Q\,dy = 0$ for every triangle $\triangle \subset D$ – domain of P, Q.

Cauchy's Theorem

18.2

$\gamma : [a,b] \longrightarrow D \subset \mathbb{C}$ continuous, piecewise cont.-diff'able curve

$t \longmapsto z(t)$

$f, g : \mathbb{C} \supset \underset{\text{open}}{D} \longrightarrow \mathbb{C}$ - complex-valued funct.

$$\int_\gamma f\,dz + g\,d\bar z := \int_a^b \left[f(z(t))\,\dot z(t) + g(z(t))\,\overline{\dot z(t)} \right] dt$$

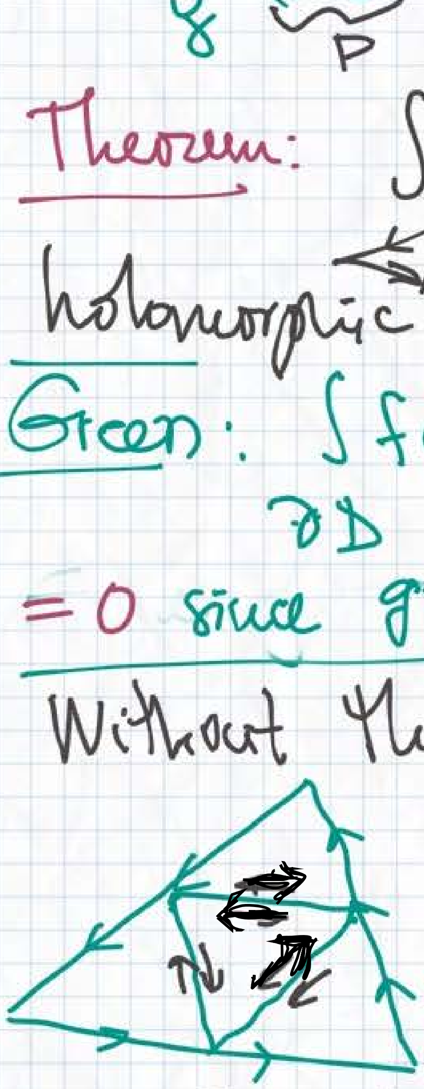
$$= \int_\gamma \underbrace{(f+g)}_{P}\,dx + i\underbrace{(f-g)}_{Q}\,dy \qquad (P, Q \text{ - complex-valued})$$

Theorem: $\int f(z)\,dz = 0$ if f is holomorphic in an open set containing Δ.

Green: $\int_{\partial D} f\,dz + g\,d\bar z = \iint_D \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar z} \right) dz \wedge d\bar z$

$= 0$ since $g = 0$ and $\partial f / \partial \bar z = 0$. assuming continuous diff. of f

Without the assumption, suppose $\int_{\partial \Delta^{(0)}} f\,dz = \alpha \neq 0$



$$\int_{\partial \Delta} f\,dz = \sum_{i=1}^{4} \int_{\partial \Delta_i} f\,dz$$

$\Rightarrow \exists\, i \in \{1,2,3,4\}$ s.t. $\left| \int_{\partial \Delta_i} f\,dz \right| \geq \frac{|\alpha|}{4}$

$\Rightarrow \exists\, \Delta \supset \Delta^{(1)} \supset \Delta^{(2)} \ldots$, s.t. diam $\Delta^{(n)} = \frac{1}{2^n}$ diam Δ

and $\left| \int_{\partial \Delta^{(n)}} f\,dz \right| \geq |\alpha| / 4^n$

~~Compactness~~ completeness: $\bigcap_{n=1}^{\infty} \Delta^{(n)} = \{z^*\}$

$$f(z) = f(z^*) + f'(z^*)(z - z^*) + o(|z - z^*|)$$

$$\int_{\partial \Delta_n} f\,dz = 0 + 0 + o\left(\frac{3\,\text{diam}^2 \Delta}{2^n \cdot 2^n} \right)$$

~~holomorphic~~ polynomial $\qquad \Delta \qquad |\Delta z| \quad |dz|$

$\xrightarrow[\text{as } n \to \infty]{} 0$ faster than $|\alpha| / 4^n$ (contradiction!)

Cauchy's Theorem and consequences

Cauchy's Theorem: $\int_{\gamma\Delta} f(z)\,dz = 0$

if $\Delta \subset D \xrightarrow{f} \mathbb{C}$ and f is holomorphic in D.

Proof: If $\int_{\gamma\Delta} = \alpha \neq 0$, find $\Delta \supset \Delta^{(1)} \supset \dots \supset \Delta^{(n)} \supset \dots$

$\left|\int_{\gamma\Delta^{(n)}}\right| \geq |\alpha|/4^n$, take $\{z^*\} = \bigcap \Delta^{(n)}$,

take $0 < \varepsilon < \frac{|\alpha|}{3(\operatorname{diam}\Delta)^2}$, find $n : \forall z \in \Delta^{(n)}$

$|f(z) - f(z^*) - f'(z^*)(z-z^*)| \leq \varepsilon |z - z^*|$,

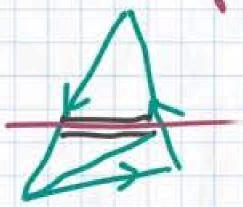
and conclude that $\left|\int_{\gamma\Delta^{(n)}}\right| \leq 3\varepsilon(\operatorname{diam}\Delta^{(n)})^2 < \frac{|\alpha|}{4^n}$

Corollary. A holomorphic function, f, locally has a holomorphic primitive, F:

$dF = f(z)\,dz + 0\,d\bar{z}$.

$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$ within disk $|z - z_0| < r$ where f is holomorphic.

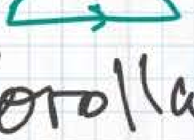
Improvement. Cauchy's thm, assuming that f is holom. in D except a straight line where f is only continuous.



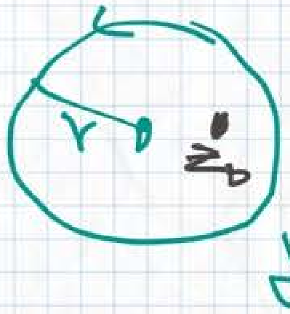
$\int f\,dz \to \int f\,dz$

continuous $\Rightarrow$ uniformly continuous, $|\varphi| \leq \varepsilon \Rightarrow \left|\int \varphi\,dt\right| \leq \varepsilon l$.

Improvement'. Cauchy's thm, assuming that f is holom. in D except one point.

$\int$  $= \int_{\Delta^{(n)}} \to 0$ as $n \to \infty$

Corollary: If $|z_0| < r$, then $\oint_{|z|=r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$

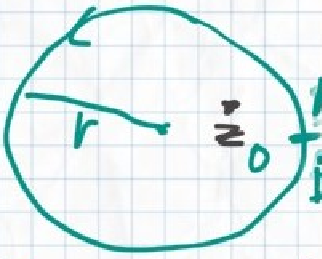


Indeed, $\frac{f(z)-f(z_0)}{z-z_0}$ is holom. for $z \neq 0$, and extends continuously to $z = z_0$ by $f'(z_0)$ $\Rightarrow$ has a primitive

Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=r} \frac{f(t)\,dt}{t-z} \quad \text{if } |z|<r$$

assuming that f is holom. in $D \supset \{|z| \le r\}$



$$\int_A^B \frac{f(z_0)}{z-z_0}\,dz = f(z_0) \underbrace{\log(z-z_0)\Big|_A^B}_{=2\pi i}$$

Remark: Here a solution to C-R equs inside the disk is represented by values $f(t)$, $|t|=r$ on the boundary. More generally $\int_\gamma \frac{\varphi(t)\,dt}{t-z}$ is holom. for z outside γ

Analyticity If f is holomorphic in $|z|<\rho$ then $f(z) = \sum_{n\ge 0} a_n z^n$ for $|z|<\rho$.

$$\frac{f(t)}{t-z} = \sum_{n\ge 0} \frac{f(t) z^n}{t^{n+1}} \quad \text{normally for } |z| \le r < |t| = r_0 < \rho$$

$$\Rightarrow f(z) = \sum_{n\ge 0} z^n \boxed{\frac{1}{2\pi i} \oint_{|t|=r_0} \frac{f(t)}{t^{n+1}}\,dt} = a_n$$

(Converges to the same f for any $r < r_0 < \rho$)

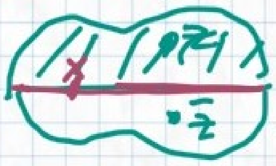
Corollaries: ① holomorphic $\Rightarrow$ analytic ($\Leftarrow$)

② holomorphic $\Rightarrow$ ∞-differentiable in complex sense.

③ holomorphic except a point (line) where it is still continuous $\Rightarrow$ holom. at this pt (on this line).

③ locally $f = g'$ $\Rightarrow$ g - ∞-differentiable $\Rightarrow$ f is (∞) differentiable $\Rightarrow$ holom.

④ Schwarz' symmetry principle $f(\mathbb{R}) \subset \mathbb{R}$



$f(\bar z) := \overline{f(z)}$

Revisiting Green's Formula

Functions $\xrightarrow{d}$ Diff. 1-forms $\xrightarrow{d}$ Diff. 2-forms

$F(x,y) \mapsto dF = F_x\,dx + F_y\,dy$

$P\,dx + Q\,dy \mapsto d(P\,dx + Q\,dy)$

$d(P\,dx + Q\,dy) = dP\wedge dx + dQ\wedge dy$

$\wedge$ – "wedge-product": $dF\wedge dG = -dG\wedge dF$

$= (P_x\,dx + P_y\,dy)\wedge dx + (Q_x\,dx + Q_y\,dy)\wedge dy$

$dx\wedge dx = -dx\wedge dx = 0$ $(= dy\wedge dy)$, $dy\wedge dx = -dx\wedge dy$

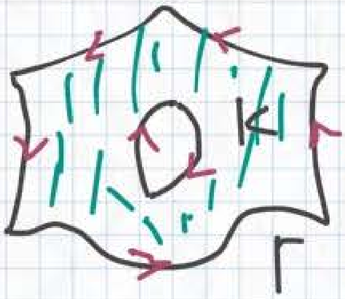
$= (Q_x - P_y)\,dx\wedge dy$

Example: $d(f\,dz + g\,d\bar z) = (g_z - f_{\bar z})\,dz\wedge d\bar z$

$dz\wedge d\bar z = (dx + i\,dy)\wedge(dx - i\,dy) = -2i\,dx\wedge dy$

Theorem ("Green-Riemann")

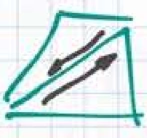
$K\subset\mathbb{R}^2$ compact region with piece-wise smooth boundary $\Gamma = \partial K$ oriented so that K "stays on the left".



P, Q – continuously differential on an open set containing K.

Then $\displaystyle\int_{\Gamma=\partial K} P\,dx + Q\,dy = \iint_K (Q_x - P_y)\,dx\wedge dy$

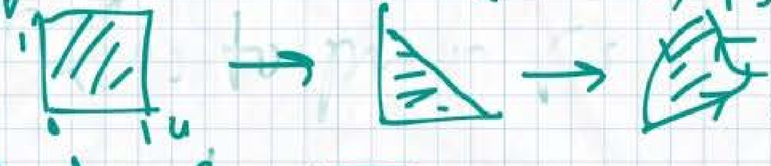
Proof. (1) Partition K into curvilinear triangles

 $\Rightarrow$ Suffices to prove for $K =$ 

(2) Invariance under changes $x(u,v)$, $y(u,v)$:

$P\,du + Q\,dv := (P x_u + Q y_u)\,du + (P x_v + Q y_v)\,dv$

$\Rightarrow (Q_u - P_v)\,du\wedge dv = (Q_x - P_y)\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv$ ← Jacobian

(3)  $\Rightarrow$ Suffices to prove for square

(4) Check for the square using the Fund. Th. of Calculus:

$\displaystyle\int_0^1\!\!\int_0^1 Q_x\,dx\wedge dy = \int_0^1 [Q(1,y) - Q(0,y)]\,dy = \int Q\,dy$

Revisiting Cauchy's Formula

Theorem. If f is holomorphic in an open set containing K, then $\int_{\Gamma = \partial K} f(z)\,dz = 0$

Proof: $\int_\Gamma f\,dz = \iint_K d\underset{\smile}{f} \wedge dz = 0$ ← differentiable

since $df \wedge dz = f'\,dz \wedge dz \equiv 0$ $\quad (f_{\bar z} = 0,\ f_z = f')$

Corollary: $f(z) = \frac{1}{2\pi i} \int_{\Gamma = \partial K} \frac{f(t)\,dt}{t - z}$ if $z \in \mathring{K}$

$= 0$ if $z \notin K$

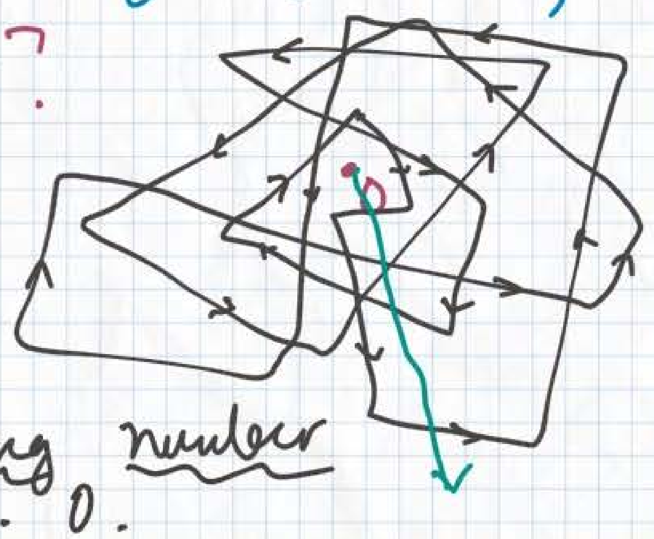
Proof: Apply Theorem to $K \setminus$ (disk around z) together with the "classical" Cauchy's formula

[Alternatively: $\int_{|t-z| = \varepsilon} \frac{f(t)\,dt}{t - z} \to 2\pi i f(z)$ as $\varepsilon \to 0$]

Remark: f is continuously differentiable $\Leftarrow$
Cauchy's Thm about $\triangle$ $\Rightarrow$ Cauchy's formula for disks $\Rightarrow$ series expansion $\Rightarrow$ analyticity

Problem: Integrals $\oint f(z)\,dz = ?$
γ - closed curve, $S^1 \to D$

Example: $\oint_\gamma \frac{dz}{z} = ?$



$\frac{dz}{z} = d\log z$
$\Rightarrow ? = 2\pi i\, N$

$I(\gamma, 0)$ ↑ winding number
↑ index of γ w.r.t. 0.

Another method: $\downarrow\ \uparrow\uparrow\ \downarrow\ \uparrow \to$ $N = +1$

Independence on the choice of ray follows from the integral definition.

Closed differential forms

functions $\xrightarrow{d}$ diff. 1-forms $\xrightarrow{d}$ diff. 2 forms

$F \longmapsto \omega \qquad \omega \longmapsto 0$

$(\omega = dF)$ exact $\Leftarrow$ closed $(d\omega = 0)$

Remark: In the book, $\omega = P\,dx + Q\,dy$ is merely continuous, so closed := locally exact

Locally exact $\not\Rightarrow$ globally exact:

$$\oint dF = F(A) - F(A) = 0, \quad \oint \frac{dz}{z} = 2\pi i N$$

$$\frac{dz}{z} = \frac{\bar{z}\,dz}{\bar{z}z} = \frac{x\,dx + y\,dy}{x^2+y^2} + i\,\frac{x\,dy - y\,dx}{x^2+y^2}$$

$z = x+iy$
$dz = dx + i\,dy$

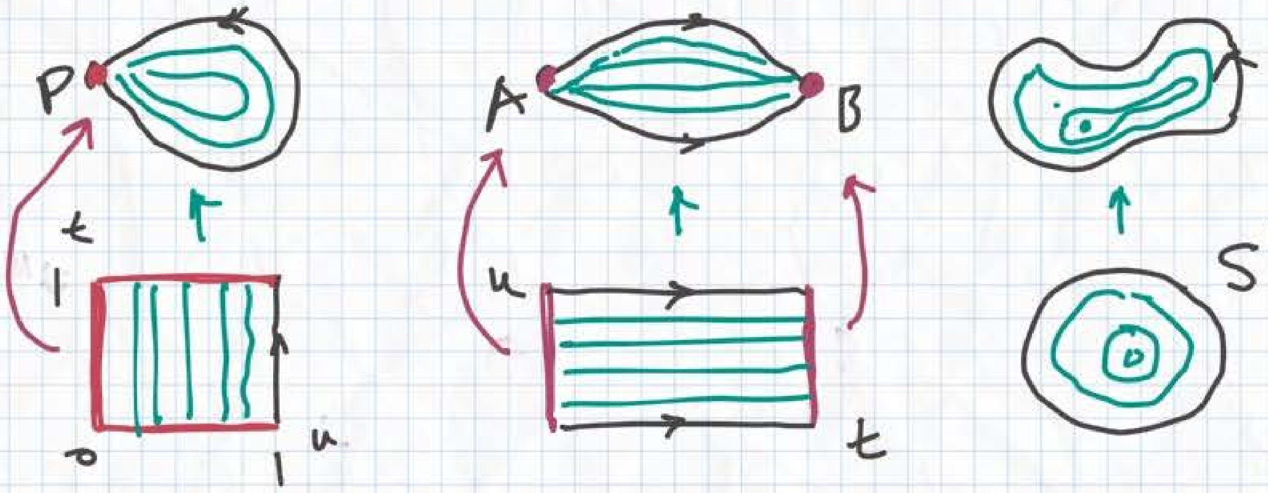
$\frac{1}{2} d \log(x^2+y^2) \qquad d \arctan \frac{y}{x}$

$$N = \frac{1}{2\pi} \oint \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi} \oint d\theta$$

polar coordinate

Theorem: If D is simply-connected, then every closed diff. 1-form in D is exact ($d\omega = 0 \Rightarrow \omega = dF$)

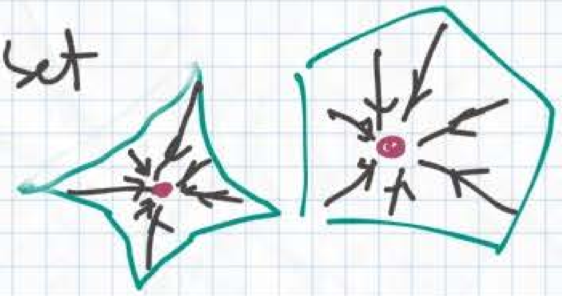
Def. Simply-connected = connected + ...



Examples of Simply-connected domains

$\mathbb{C}$, $\mathbb{U} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ ← unit disk

Any convex open set
Any star-shaped
Sphere S^2



Proof of Theorem.

Main Idea: Construct $F = \int_{P_0}^{P} \omega$ and prove path-independence.

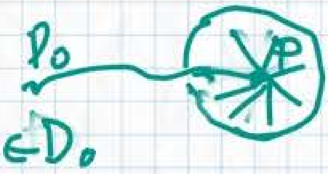
① A connected open D is path-connected

$P_0 \in D_0 = \{ P \in D \mid P \text{ can be connected to } P_0 \}$

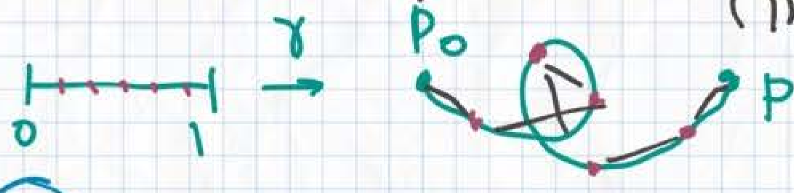
open and closed $\Rightarrow D_0 = D$

$P \in D_0 \Rightarrow B_\varepsilon(P) \subset D_0$

$D_0 \ni P_n \to P,\ P_{n_0} \in B_\varepsilon(P) \Rightarrow P \in D_0$

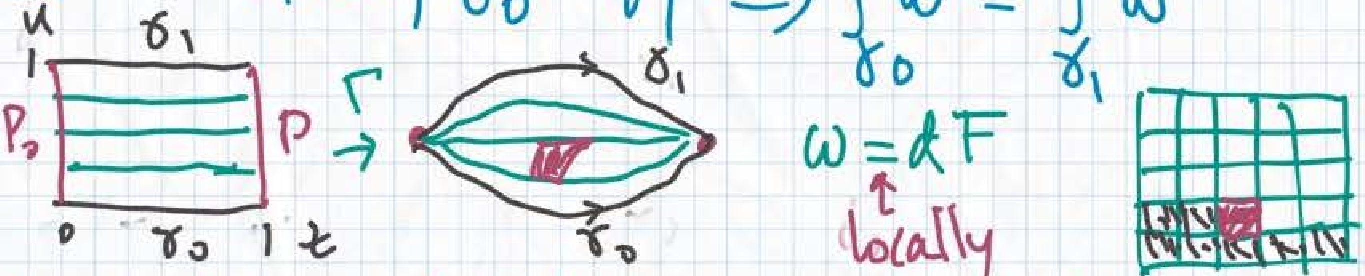


② Continuous path $\rightsquigarrow$ piece-wise smooth path (in fact piece-wise linear)



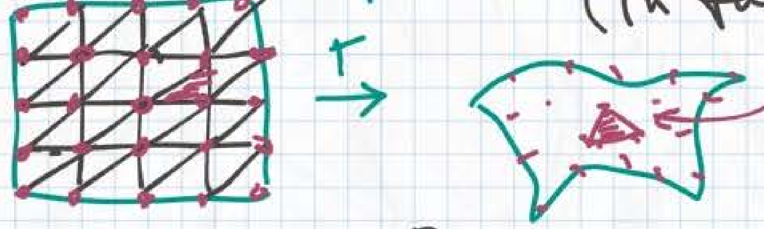
compactness of $[0,1]$
uniform cont.ity of γ
openness of D.

③ ω-closed, $\gamma_0 \sim \gamma_1 \Rightarrow \int_{\gamma_0} \omega = \int_{\gamma_1} \omega$



compactness of $[0,1] \times [0,1]$, uniform cont. of Γ.

④ $\gamma_0 \sim \gamma_1 \rightsquigarrow$ piece-wise smooth homotopy (in fact piece-wise linear)



linear interpolation
(compactness of $[0,1] \times [0,1]$, unif. cont. of Γ, openness of D)

⑤ $F(P) := \int_{P_0}^{P} \omega$ – does not depend on path connecting P_0 with P

provided that ω is closed, D - simply-connected

$\Rightarrow F =$ local primitives up to const. $\Rightarrow dF = \omega$

Corollaries. If $f: D \to \mathbb{C}$ is holomorphic then $\oint_{\gamma_0} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz$ when $\gamma_0 \sim \gamma_1$;

$\oint_\gamma f(z)\,dz = 0$ if D is simply connected

Cauchy's Inequalities

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=r_0} \frac{f(t)\,dt}{t-z} \qquad |z| < r_0 < \rho$$

holomorphic in $|z|<\rho$

$$\Rightarrow \quad f(z) = \sum_{n\geq 0} z^n \underbrace{\left[\frac{1}{2\pi i} \oint_{|t|=r_0} \frac{f(t)}{t^{n+1}}\,dt\right]}_{a_n \text{ - independent of } r_0}$$

$|z| < r_0 < \rho$

(homotopy? Green?)

$\Rightarrow$

- $\boxed{|a_n| \leq \frac{M(r)}{r^n}}$, $r<\rho$, $M(r) := \max_{|t|=r} |f(t)|$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, r^{-n} e^{-in\theta}\,d\theta$$

- Convergence radius $\geq \rho$
- $f = \sum a_n z^n$ cannot be analytically continued to a disk larger than the convergence disk of the series.

Examples: $\frac{1}{1+x^2} = 1 - x^2 + x^4 + \ldots \quad x \neq \pm i$

$$\frac{z}{(1-z/z_1)(1-z/z_2)} \quad \rho = \min(|z_1|, |z_2|)$$

Def. Entire func. := holom. $f: \mathbb{C} \to \mathbb{C}$

Examples: polyn, e^z, $\sin z$, $\cosh z$

$f(z) = \sum_{n\geq 0} a_n z^n$, $\limsup_{n\to\infty} |a_n|^{1/n} = 0$

Liouville's Thm Bounded entire functions are constant.

Proof: $|f(z)| \leq M$ for all $z \in \mathbb{C}$

$\Rightarrow |a_n| \leq \frac{M}{r^n}$ for any $r>0$

$\Rightarrow a_n = 0$ for all $n \geq 1$.

The Fundamental Thm of Algebra

2nd proof (the 1st proof in hw 4)

Suppose polyn $P \not\equiv$ const has no roots.

$$\frac{1}{P(z)} = \frac{1}{z^n} \frac{1}{[a_0 + \frac{a_1}{z} + \dots + \frac{a_n}{z^n}]} \to 0 \quad \text{as } |z| \to \infty$$

holom. in $\mathbb{C}$ and bounded $\Rightarrow P \equiv$ const

Characterization of rational functions

$\frac{P(z)}{Q(z)} \Leftrightarrow$ Functions f meromorphic in $\mathbb{C}$ with polyn. growth at ∞

$$|f(z)| \le M |z|^D \text{ for } |z| \ge R.$$

$\Rightarrow$ f has no poles $|z| \ge R$

$\Rightarrow$ finitely many poles z_i, $|z_i| < R$

$\Rightarrow$ $g(z) := f(z)(z-z_1)^{m_1} \dots (z-z_N)^{m_N}$ – entire

(m_i – order of pole z_i)

$$\Rightarrow g(z) = \sum_{n \ge 0} a_n z^n$$

$$|g(z)| \le \tilde{M} |z|^{D+m_1+\dots+m_N} \qquad |z| \ge R.$$

$$\Rightarrow |a_n| \le \frac{\tilde{M} r^{D+m_1+\dots+m_N}}{r^n} \qquad r \ge R$$

$\Rightarrow$ $a_n = 0$ for $n > D + m_1 + \dots + m_N$

$$\Rightarrow f(z) = \frac{a_0 + a_1 z + \dots + a_{D+\Sigma m_i} z^{D+\Sigma m_i}}{(z-z_1)^{m_1} \dots (z-z_N)^{m_N}}$$

Corollary: If f is bounded at ∞, then it has a limit at ∞.

$$\deg P \le \deg Q \Rightarrow \frac{p_0 z^m + \dots}{q_0 z^m + \dots} \xrightarrow[|z| \to \infty]{} \frac{p_0}{q_0}, \quad 0 \ne q_0$$

The Mean Value Property

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta$$

$\Leftrightarrow$ $f(z)$ = average value of $f(t)$ along the circle $|t-z| = r$

holom (in a disk of radius $> r$ centered at z)

Theorem (Maximum Modulus Principle).
$f: D \to \mathbb{C}$ - continuous, satisfying MVP.
Suppose $|f|$ has a local max. at $a \in D$.
Then $f \equiv f(a)$ in a neighborhood of a.

Proof: WLOG, $\mathbb{R} \ni f(a) \geq M(r) := \max\limits_{|z-a|=r} |f(z)|$

$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$

$\Rightarrow f(a) \leq M(r) \Rightarrow M(r) = f(a) \Rightarrow$

$g(z) := f(a) - \operatorname{Re} f(z)$ $\left(\geq f(a) - |f(z)|\right) \geq 0$ in a nbhd of a.

$= 0$ only if $(\operatorname{Im} f(z) = 0)$ i.e. $f(z) = f(a)$.

$\frac{1}{2\pi}\int_0^{2\pi} g(a+re^{i\theta})\, d\theta = g(a) = 0 \Rightarrow g \equiv 0$ in that nbhd

$\Rightarrow f(z) = f(a)$ — ⌞ —

Corollary. D - connected, bounded (open),
$f: \overline{D} \to \mathbb{C}$ - continuous, satisfies MVP in D.
Then $|f(z)| \leq M := \max\limits_{t \in \overline{D} \setminus D} |f(t)|$

and if "=" at some $z_0 \in D$, then $f \equiv$ const.

Proof: $D_0 = \{a \in D \mid |f(a)| = \max\limits_{z \in \overline{D}} |f(z)|\}$

- closed in D since $|f|$ - continuous
- open in D by the MMP (Theorem)

$\Rightarrow D_0 = D$ (i.e. $f \equiv$ const) or $D_0 = \emptyset$ (i.e. $|f|\big|_D < M$)

Theorem (Schwarz' lemma) 13.2

$\mathbb{U} := \{ z \in \mathbb{C} \mid |z| < 1 \}$

$f : \mathbb{U} \to \mathbb{U}$ - holomorphic, $f(0) = 0$

Then $|f(z)| \leq |z|$ for all $z \in \mathbb{U}$, and if "=" for some $z_0 \neq 0$, then $f(z) = e^{i\theta} z$

Proof: $g(z) := f(z)/z$ is holom. in $\mathbb{U}$ and $|g(z)| \leq \frac{1}{r}$ for $|z| = \cancel{r} \leq r$ (MMP)

Take $\lim$ as $r \to 1^-$: $|g(z)| \leq 1$.

If $|g(z_0)| = 1$, then $g(z) \equiv g(z_0) (= e^{i\theta})$

Corollary 1 $|f'(0)| \leq 1$, and if "=", the $f(z) = e^{i\theta} z$

Indeed, $|g(0)| \leq 1$, and if "=", then $g(z) = e^{i\theta}$

Corollary 2 If $f : \mathbb{U} \to \mathbb{U}$, $f(0) = 0$, is invertible, then $f(z) = e^{i\theta} z$.

Indeed, for $h = f^{-1}$, $h'(0) = 1/f'(0)$

Corollary 3. $\mathrm{Aut}(\mathbb{U}) = \left\{ e^{i\theta} \frac{z-a}{1-\bar{a}z}, |a| < 1 \right\}$

1° $w = e^{i\theta} \frac{z-a}{1-\bar{a}z} \Leftrightarrow z = e^{-i\theta} \frac{w + e^{i\theta} a}{1 + e^{-i\theta} \bar{a} w}$

Exercise!

are inverse transf. $\mathbb{U} \to \mathbb{U}$, and form a group w.r.t. composition.

2° $h \in \mathrm{Aut}(\mathbb{U})$, $h(0) = a \Rightarrow$

$w = \frac{h(z) - a}{1 - \bar{a} h(z)} =: f \in \mathrm{Aut}(\mathbb{U})$, $f(0) = 0$

$\Rightarrow f(z) = e^{i\theta} z \Rightarrow h^{-1}(w) = e^{-i\theta} \frac{w - a}{1 - \bar{a} w}$

Corollary 4. Automorphisms of $\mathbb{U}$ are fractional-linear, $w = \frac{az+b}{cz+d}$, extend to maps of $\partial\mathbb{U} \to \partial\mathbb{U}$, and form a 3-parametric group.

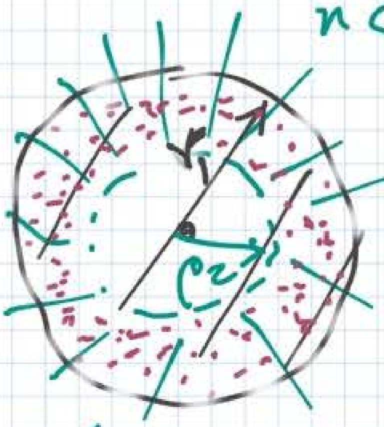
Laurent series

(14.1

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n\geq 0} a_n z^n + \sum_{n<0} a_n z^n$$

converges in $|z|<\rho_1$ — "principal part"

"principal part" $\displaystyle\sum_{n<0} a_n z^n = \sum_{n>0} a_{-n} w^n$ converges in $|w|<\frac{1}{\rho_2}$



$|z|>\rho_2$

$|z|<\rho_1$

A Laurent series converges in an annulus $0\leq\rho_2<|z|<\rho_1\leq\infty$

Theorem: A function f holomorphic in an annulus $\rho_2<|z|<\rho_1$ expands in it into a (unique) Laurent series converging normally in any $\rho_2<r_2\leq|z|\leq r_1<\rho_1$

1° Cauchy's formula:

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t-z} = \frac{1}{2\pi i}\oint_{|t|=r_1'>r_1}\frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\oint_{|t|=r_2'<r_2}\frac{f(t)\,dt}{t-z}$$

γ 

$r_2<|z|<r_1$

2° Geometric series:

$$\frac{1}{t-z}=\sum_{n\geq 0}\frac{z^n}{t^{n+1}} \qquad \frac{1}{t-z}=-\sum_{n>0}\frac{t^{n-1}}{z^n}$$

$|z|<|t|=r_1'>r_1$ $\qquad$ $|z|>|t|=r_2'<r_2$

3° Termwise Integration:

$$f(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n=\pm\frac{1}{2\pi i}\oint\frac{f(t)\,dt}{t^{n+1}} \quad \begin{matrix} n\geq 0 \\ n<0 \end{matrix}$$

$\rho_2<|t|=r<\rho_1$

does not depend on r (homotopy argument)

4° Convergence

O, normal in $\rho_2<r_2'<r_2\leq|z|\leq r_1<r_1'<\rho_1$

In 2°, $\displaystyle\sum_{n\geq 0}\left(\frac{r_1}{r_1'}\right)^n+\sum_{n>0}\left(\frac{r_2'}{r_2}\right)^n<\infty$

5° Uniqueness: $f(re^{i\theta})=\sum a_n r^n e^{in\theta}$ — Fourier coeff.

Example. $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$= \sum_{n\ge 0} z^n \left(1 - \frac{1}{2^{n+1}}\right)$, $|z|<1$

$= \sum_{n>0} \frac{1}{z^n}\left(2^{n-1}-1\right)$, $|z|>2$

$= -\sum_{n\ge 0} \frac{z^n}{2^{n+1}} - \sum_{n<0} z^n = f_+(z) + f_-(z)$, $1<|z|<2$

Decomposition $f(z) = f_+(z) + f_-(z)$

holomorphic: $\rho_2 < |z| < \rho_1$, $|z|<\rho_1$, $|z|>\rho_2$

unique if $f_- \to 0$ as $|z| \to \infty$.

Proof: $f = \underbrace{\sum_{n\ge 0} a_n z^n}_{f_+} + \underbrace{\sum_{n<0} a_n z^n}_{f_-} = g_+ + g_-$

$f_+ - g_+ = g_- - f_- \equiv 0$ Liouville!

Cauchy's inequalities:

$$a_n = \frac{1}{2\pi} \oint_{|z|=r} f(re^{i\theta})\, r^{-n} e^{-in\theta}\, d\theta$$

$\Longrightarrow |a_n| \le M(r)/r^n$, $M(r) = \max |f(re^{i\theta})|$

$n = 0, \pm 1, \pm 2, \dots$

Isolated Singularities: $0<|z|<\rho$

1° If $|f| < M \Longleftrightarrow f$ is holom. at $z=0$

$|a_n| \le M/r^n$, $r \to 0 \Rightarrow a_n = 0$ for $n<0$.

2° Finitely many of $a_{-1}, a_{-2}, \dots$ are $\ne 0$

$\Longleftrightarrow f$ has a pole at $z=0$.

3° Infinitely many of $a_{-1}, a_{-2}, \dots \ne 0$

$\Longleftrightarrow$ essential singularity Example: $e^{1/z}$

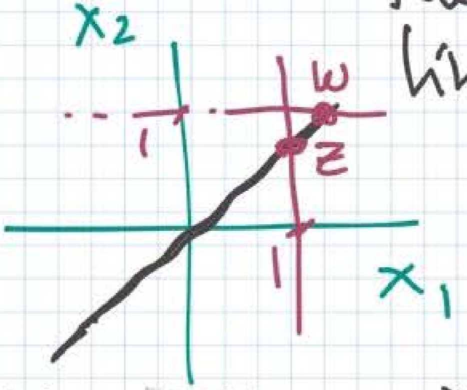
Weierstrass' Thm: $f(0<|z|<\varepsilon)$ is dense in $\mathbb{C}$

If $|f(z)-a| \ge r \Rightarrow g(z) = \frac{1}{f(z)-a}$ is bounded

$\Rightarrow$ holom. at $z=0 \Rightarrow f(z) = a + \frac{1}{g(z)}$ is merom.

The Riemann Sphere

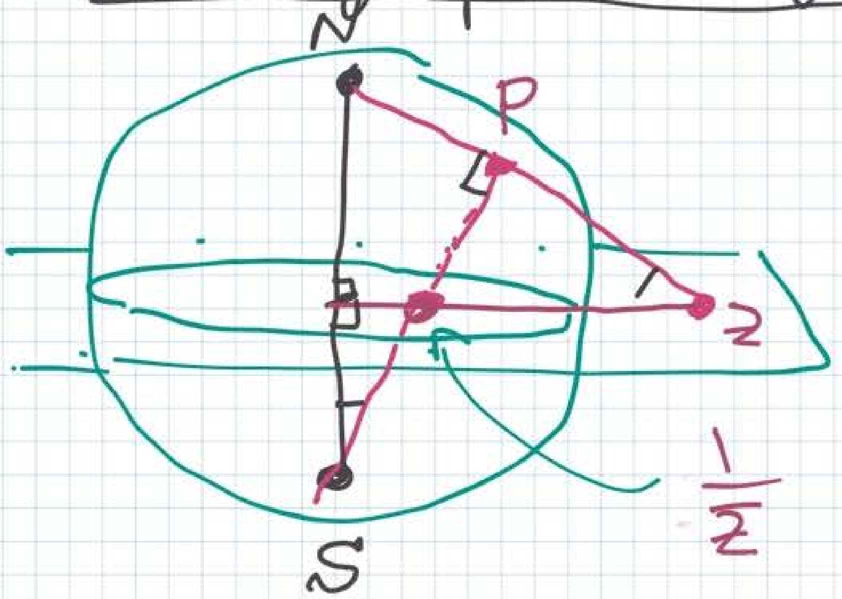
$\mathbb{C}P^1$ - the complex projective line
= the set of 1-dimensional linear subspaces in $\mathbb{C}^2$



z = the complex "slope" $\in \mathbb{C} \cup \infty$

$x_2 = z x_1 \qquad x_1 = w x_2 \qquad w = \frac{1}{z}$

Stereographic Projection



$\mathbb{C}P^1 \simeq S^2$

Rewriting meromorphic functions:

$f : \mathbb{D} \longrightarrow \mathbb{C}P^1$

$\uparrow$ holomorphic maps

Near a pole, $f(z) = \frac{g(z)}{(z-z_0)^k}$, $g(z_0) \neq 0$, $k > 0$

$\Rightarrow f(z_0) = \infty$, $\frac{1}{f(z)} = (z-z_0)^k \frac{1}{g(z)}$ holom. near z_0.

! At essential singularities, $\lim_{z \to z_0} f(z) \neq \infty$ (does not exist)

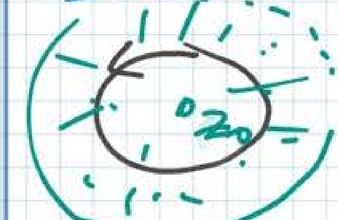
Automorphisms of $\mathbb{C}P^1$

$GL_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, ad \neq bc \right\} \Big/ \left[\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix} \right]$

$h(\infty) = z_0 \Rightarrow \frac{1}{h(z) - z_0} = cz + d \Rightarrow h(z) = \frac{az+b}{cz+d}$

Residues

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$



$0<|z-z_0|<\rho$

$$\oint_{|z-z_0|=r} f(z)\,dz = 2\pi i\, a_{-1}$$

residue of $f(z)dz$ at z_0

$$f(z)dz = d\left\{\sum_{n\neq -1} a_n \frac{(z-z_0)^{n+1}}{n+1}\right\} + a_{-1}\, d\log(z-z_0)$$

Residue at ∞

$$f(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \quad \rho<|z|<\infty$$

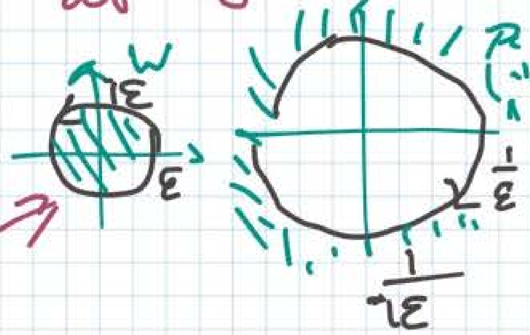
$$f(z)dz = -f\left(\frac{1}{w}\right)\frac{dw}{w^2} = -\sum a_{-2-n} w^n dw$$

$$=\left[\dots - \frac{a_1}{w^3} - \frac{a_0}{w^2} - \frac{a_{-1}}{w} - a_{-2} - a_{-3} w - \dots\right]dw$$

Residue of $f(z)dz$ at $z_0=\infty$

$$\oint f(z)\,dz = 2\pi i\,(-a_{-1})$$

$|w|=\varepsilon$ ← counter-clockwise



The Residue Theorem

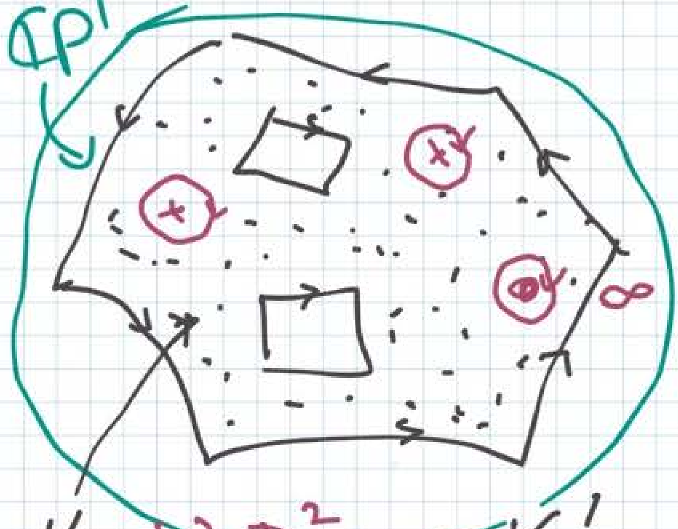
$$\int_{\partial K} f(z)\,dz = 2\pi i\left(\sum_k \operatorname{Res}_{z_k} f\,dz\right)$$

z_k – singularities inside K

$f(z)$ – holom. in $D-\{$isolated singularities$\}$

K – compact in D; D – open in $\mathbb{C}P^1$

with piecewise diff. boundary ∂K avoiding singularities of f.



$$\int_{\partial K'} f(z)\,dz = 0$$

$$\Downarrow$$

$$\int_{\partial K} f(z)\,dz = \sum_k \int_{\partial D_k^2} f(z)\,dz$$

$$= \sum_k 2\pi i \operatorname{Res}_{z_k} f\,dz$$

$$K - \bigsqcup_k D_k^2 = K'$$

Corollary. f – holom. in $\mathbb{C}P^1 - \{$isolated singularities$\}$

$$\Rightarrow \sum_k \operatorname{Res}_{z_k} f\,dz = 0 \qquad \partial K = \emptyset$$

Logarithmic derivatives

$$d\log f = df/f = \frac{f'}{f}\,dz$$

$f=(z-z_0)^k g(z)$, $g(z_0)\neq 0$, $k\in\mathbb{Z}$

$\Rightarrow \mathrm{Res}_{z_0}\dfrac{df}{f} = k$ (1-st order pole)

$$d\log f = \Big[\frac{k}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{holomorphic}}\Big]dz$$

Theorem $\dfrac{1}{2\pi i}\displaystyle\oint_{\gamma K}\frac{f'dz}{f} = \#\text{Zeros} - \#\text{Poles}$ inside K, counting with multiplicities

Corollary: f - degree n polynomial $\Rightarrow \#\text{Zeros} = n$.

$f = z^n\big(\underset{\neq 0}{a_0} + \frac{a_1}{z} + \dots\big) = \frac{1}{w^n}\big(\underset{\neq 0}{a_0} + a_1 w + \dots\big)$, $w=\frac{1}{z}$

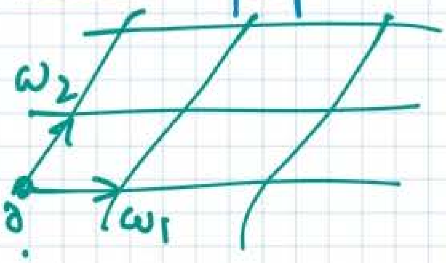
n-th order pole at $z=\infty$

But: Total $\sum \mathrm{Res} = 0$.

Corollary. f-rational function, $\mathbb{CP}^1 \to \mathbb{CP}^1$

$\Rightarrow \#\text{Zeros} = \#\text{Poles}$ (including at $z=\infty$)

Doubly-periodic meromorphic functions



$\Omega = \{m\omega_1 + n\omega_2\}$, $f(z+\omega) = f(z)$

$f: \mathbb{C}/\Omega \to \mathbb{CP}^1$

$\simeq S^1\times S^1$ - torus

1° $\sum \mathrm{Res}_{z_i} f\,dz = \frac{1}{2\pi i}\oint f\,dz = 0$ periodicity!

2° $\#Z = \#P$ (apply 1° to $\frac{f'}{f}$)

3° $\sum \alpha_i \equiv \sum \beta_i \mod \Omega$ [The same $f(\alpha_i) = \text{const}$]

$\frac{1}{2\pi i}\oint z\frac{f'(z)\,dz}{f(z)} = \sum\alpha_i - \sum\beta_i$ $\left(\sum \mathrm{Res}\, \frac{z\,df}{f}\right)$

$= \frac{\omega_1}{2\pi i}\displaystyle\int_{\gamma_2}\frac{df}{f} - \frac{\omega_2}{2\pi i}\int_{\gamma_1}\frac{df}{f} \in \Omega$

integers.

Example $\zeta(2n)$ and Bernoulli numbers

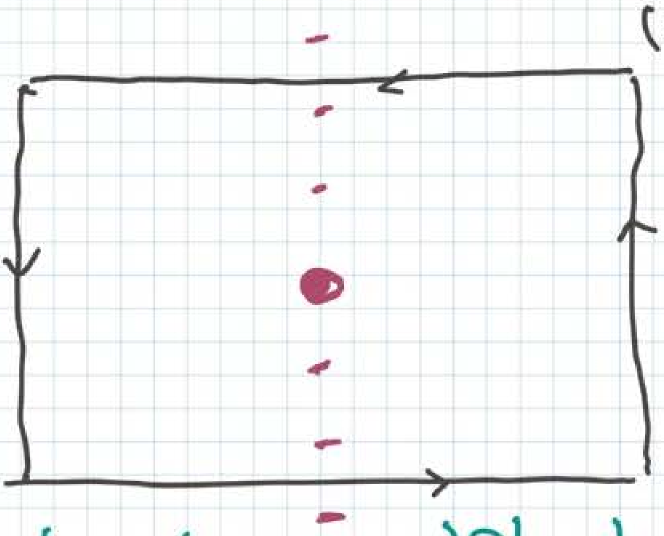
$$\sum_{m>0} \frac{1}{m^2} = \frac{\pi^2}{6} \qquad \sum_{m>0} \frac{1}{m^{2n}} = ?$$

$$f = \frac{1}{z^{2n}(1-e^{-z})}$$

$2n+1$st ~~th~~ order pole at 0

1st order poles at $2\pi i m$, $m \neq 0$

$(\pi(2m+1), \pi(2m+1)i)$



$$\boxed{\oint_{\square} f(z)\,dz \to 0 \quad \text{as } m \to \infty}$$

$z = t \pm \pi(2m+1)i \qquad |f| = \frac{1}{|z|^{2n}} \frac{1}{1+e^{-t}} < \frac{1}{[\pi(2m+1)]^{2n}}$

$z = \pi(2m+1) \pm it \qquad |f| = \frac{1}{|z|^{2n}} \frac{1}{\left|1 - \frac{e^{\mp it}}{e^{\pi(2m+1)}}\right|} < \frac{2}{[\pi(2m+1)]^{2n}}$

$z = -\pi(2m+1) \pm it \qquad |f| = \frac{1}{|z|^{2n}} \frac{1}{|e^{\pi(2m+1)} - e^{\pm it}|} < \frac{1}{[\pi(2m+1)]^{2n}}$

$$\Rightarrow \operatorname{Res}_{z=0} f\,dz = -\sum_{m=-\infty}^{\infty} \operatorname{Res}_{2\pi i m} f\,dz = -\sum_{m \neq 0} \frac{1}{(2\pi i m)^{2n}}$$

$$\frac{dz}{1-e^{-z+2\pi i m}} = \frac{dz}{1-1+z+\dots} = \frac{dz}{z}+\dots \qquad = \frac{2(-1)^{n-1}}{(2\pi)^{2n}} \zeta(2n)$$

Bernoulli numbers:

$$\frac{z}{1-e^{-z}} = \frac{z}{2} + \underbrace{\frac{z}{2}\,\frac{1+e^{-z}}{1-e^{-z}}}_{\text{even}} = 1 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{(2n)!} z^{2n}$$

Bernoulli #.

$$\frac{1 + \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{1}{24}\left(\frac{z}{2}\right)^4 + \dots}{1 + \frac{1}{6}\left(\frac{z}{2}\right)^2 + \frac{1}{120}\left(\frac{z}{2}\right)^4} = \left(1 + \frac{z^2}{8} + \frac{z^4}{16 \cdot 24} + \dots\right) \times \left(1 - \frac{z^2}{24} - \frac{z^4}{120 \cdot 16} + \frac{z^4}{16 \cdot 36} + \dots\right)$$

$$= 1 + \frac{z^2}{12} - \frac{z^4}{720} + \dots \qquad B_2 = \frac{1}{6}, \quad B_4 = 1/30$$

$$\boxed{\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}} \qquad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}$$

Evaluation of definite integrals

The Fund. Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b)-F(a) \quad \text{where } F'=f.$$

Example: $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx =: I = \sqrt{2\pi}$

$$I^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^\infty e^{-r^2/2}\,\cancel{r\,dr}\,\frac{dr^2}{2}\int_0^{2\pi} d\theta$$

Using Residues: Type 1.

$$I := \int_0^{2\pi} R(\cos t, \sin t)\,dt$$

$R(x,y)$ · rational without poles on $x^2+y^2=1$

$z = e^{it}$, $dz = iz\,dt$

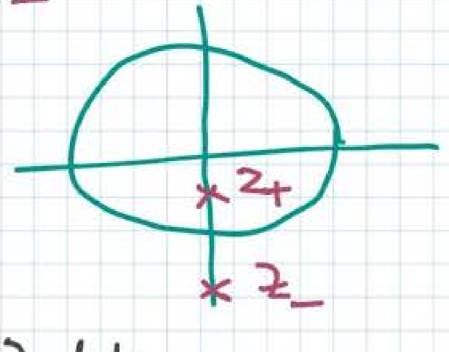
$$I = \int_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

Example: $\int_0^{2\pi}\frac{dt}{a+\sin t} = \oint_{|z|=1}\frac{2\,dz}{\underbrace{(z^2+2aiz-1)}_{=0}}$, $a>1$

$= 4\pi i \operatorname{Res}_{z_+}\frac{dz}{(z-z_+)(z-z_-)}$ $\qquad z_\pm = -ai \pm \sqrt{-a^2+1}$

$= 4\pi i/(z_+-z_-)$

$= 4\pi i / 2i\sqrt{a^2-1} = \frac{2\pi}{\sqrt{a^2-1}}$



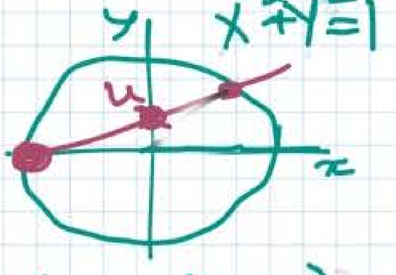
Remark: $\int R(\cos t, \sin t)\,dt$

$= \int R\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)\frac{2du}{1+u^2}$

$\uparrow$

$t = 2\arctan u$

$dt = \frac{2du}{1+u^2}$



$y = u(1+x)$

$u^2(1+x)^2+x^2=1$

$x^2(1+u^2)+2xu^2+u^2-1=0$

$x=-1, \begin{cases} x = \frac{1-u^2}{1+u^2} \\ y = \frac{2u}{1+u^2}\end{cases}$

= (decompose into partial fractions)

Exercise

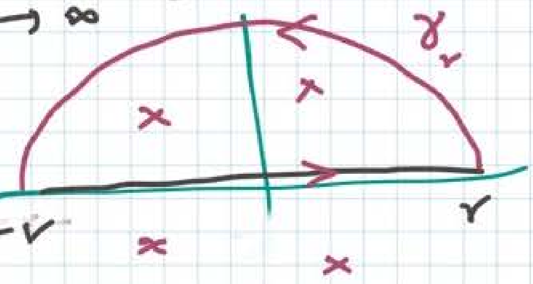
$= \frac{2}{\sqrt{a^2-1}}\arctan\frac{au+1}{\sqrt{a^2-1}}\Big|$

Using Residues: Type 2.

$$I := \int_{-\infty}^{\infty} R(x)\,dx$$

R - rational $= \frac{P(x)}{Q(x)}$
no real poles
$\deg Q - \deg P \geq 2$

$$= \lim_{r\to\infty}\int_{-r}^{r} R(x)\,dx = \lim_{r\to\infty}\oint_{\gamma_r} R(z)\,dz$$



$$i\int_0^{\pi} \underbrace{R(re^{it})\,re^{it}}_{\to 0 \text{ as } r\to\infty}\,dt$$

$$= 2\pi i \sum_{\operatorname{Im} z_k>0} \operatorname{Res}_{z_k} R(z)\,dz$$

Example $I := \int_0^{\infty}\frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}$

$$\frac{dz}{(1+z^2)^2} = \frac{dz}{(z-i)^2(z+i)^2}$$

Lemma $\operatorname{Res}_{z_0}\frac{g(z)}{(z-z_0)^{k+1}} = \frac{g^{(k)}(z_0)}{k!}$ (g holom.)

$$g(z) = \dots + \frac{g^{(k)}(z_0)}{k!}(z-z_0)^k + \dots$$

Corollary: 1-st order pole $\Rightarrow$ $\operatorname{Res} = g(z_0)$
2-nd order pole $\Rightarrow$ $\operatorname{Res} = g'(z_0)$

$$\frac{d}{dz}\Big|_{z=i}\frac{1}{(z+i)^2} = \frac{-2}{(z+i)^3}\Big|_{z=i} = \frac{1}{4i} \Rightarrow I = \frac{1}{2}\,\frac{2\pi i}{4i} = \pi/4$$

Remark: $\int \frac{dx}{1+x^2} = \frac{x}{1+x^2}\Big| + \int\frac{x\cdot 2x}{(1+x^2)^2}\,dx$

$$= \frac{x}{1+x^2}\Big| + \underline{2\int\frac{dx}{1+x^2}} - 2\int\frac{dx}{(1+x^2)^2} \Rightarrow$$

$$\int\frac{dx}{(1+x^2)^2} = \frac{1}{2}\,\frac{x}{1+x^2}\Big| + \frac{1}{2}\int\frac{dx}{1+x^2} = \frac{1}{2}\arctan x\Big|$$

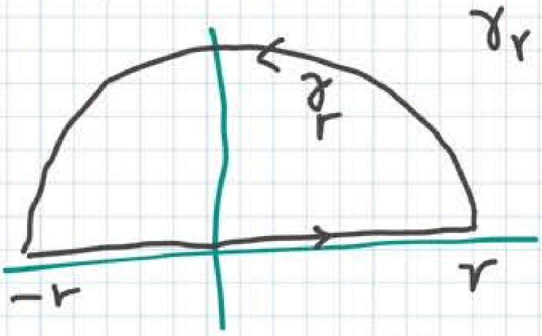
$$\Rightarrow \int_0^{\infty}\frac{dx}{(1+x^2)^2} = 0 - 0 + \frac{1}{2}\,\frac{\pi}{2}$$

Integration Using Residues, Type 3

$$I := \int_{-\infty}^{\infty} \underbrace{R(x)}_{\text{rational function, } |R(z)| \to 0 \text{ as } |z| \to \infty} \underbrace{e^{ix}}_{\cos x + i\sin x} dx$$

Type 3a: R has no real poles

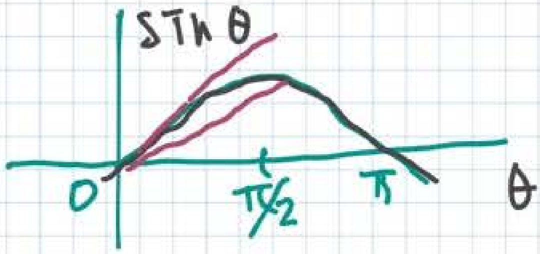
Lemma: $\int_{\gamma_r} R(z) e^{iz} dz \to 0$ as $r \to \infty$.



$$\left| \int_0^{\pi} R(re^{i\theta}) e^{ir\cos\theta - r\sin\theta} \, i re^{i\theta} d\theta \right|$$

$$\leq M(r) \int_0^{\pi} e^{-r\sin\theta} r \, d\theta$$

$M(r) := \max_{|z|=r} R(z)$



$\frac{2\theta}{\pi} \leq \sin\theta \leq \theta$ for $0 \leq \theta \leq \pi/2$

$$\int_0^{\pi} e^{-r\sin\theta} r \, d\theta = 2 \int_0^{\pi/2} e^{-r\sin\theta} r \, d\theta$$

$$\leq 2 \int_0^{\pi/2} e^{-2r\theta/\pi} r \, d\theta$$

$$\leq 2\frac{\pi}{2} \int_0^{\infty} e^{-2r\theta/\pi} d\, r\theta \frac{2}{\pi} = \pi$$

Corollary. $I = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}_{z_k} R(z) e^{iz} dz$

Example $I := \int_{-\infty}^{\infty} \frac{\cos^2 x}{1+x^2} dx \left[= \int_{-\infty}^{\infty} \frac{(1+\cos 2x)}{2(1+x^2)} dx\right]$

$$\cos^2 z = \frac{e^{2iz} + 2 + e^{-2iz}}{4}$$

$$\int_{-\infty}^{\infty} \frac{dx}{2(1+x^2)} = \frac{1}{2} \arctan x \Big|_{-\infty}^{\infty} = \pi/2$$

$$\int_{-\infty}^{\infty} \frac{e^{2ix} dx}{1+x^2} = 2\pi i \operatorname{Res}_{z=i} \frac{e^{2iz} dz}{1+z^2} = \frac{2\pi i}{2i} e^{-2}$$

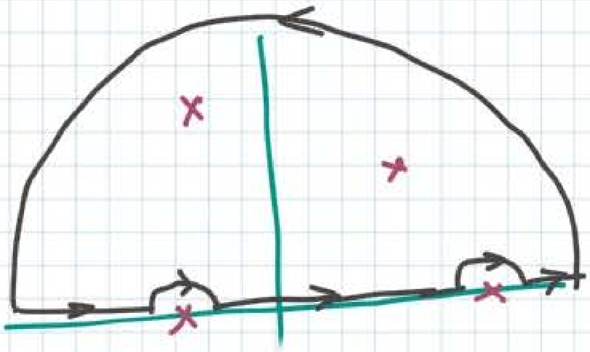
$$\int_{-\infty}^{\infty} \frac{e^{-2ix} dx}{1+x^2} = -2\pi i \operatorname{Res}_{z=-i} \frac{e^{-2iz} dz}{1+z^2} = \frac{-2\pi i}{-2i} e^{-2}$$

$$I = \frac{\pi}{2}(1 + e^{-2})$$

Integration Using Residues. Type 3(b) | 18.2

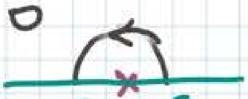
$$I=\int_{-\infty}^{\infty} R(x)e^{ix}\,dx, \quad R(z)\to 0 \text{ as } |z|\to\infty,$$

$R(x)$ is allowed simple poles on the real axis.



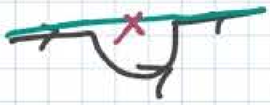
Lemma: g - holom. at $z=0$

$$\lim_{\varepsilon\to 0}\int \frac{g(z)}{z}\,dz = \pi i\, g(0)$$

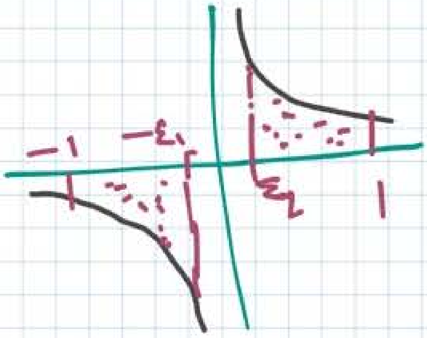


Proof: $$\underbrace{\int_{\gamma_\varepsilon}\frac{g(0)\,dz}{z}}_{=\,g(0)(\pi i)} + \underbrace{\int_{\gamma_\varepsilon} h(z)\,dz}_{\int_0^{\pi} h(\varepsilon e^{i\theta})\,\varepsilon\, e^{i\theta}\, i\,d\theta}$$

h – holom. at $z=0$; $\varepsilon \to 0$

Question: What if we use  ?

Remark:



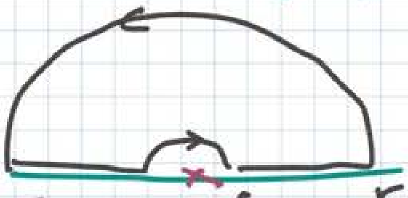
$$\lim_{\varepsilon_1,\varepsilon_2\to 0^+}\int\frac{dx}{x} = \text{anything}$$

$$\lim_{\varepsilon_1=\varepsilon_2\to 0}\int\frac{dx}{x} = 0$$

Example: $$I := \int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$$

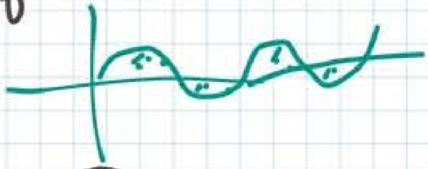
$$= \frac{1}{2}\operatorname{Im}\lim_{\substack{\varepsilon\to 0\\ r\to\infty}}\left[\int_{-r}^{-\varepsilon}\frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{r}\frac{e^{ix}}{x}\,dx\right]$$

$$= \frac{1}{2}\operatorname{Im}\pi i = \frac{\pi}{2}$$



Remark: $\int_0^{\infty}\frac{\sin x}{x}\,dx$ converges, but not absolutely

$\sum(-1)^k a_k$ converges if $a_k\to 0^+$.



Proposition: In general

$$I = 2\pi i\sum_{\operatorname{Im} z_k>0}\operatorname{Res}_{z_k} R(z)e^{iz}dz + \pi i\sum_{\operatorname{Im} z_k=0}\operatorname{Res}_{z_k} R(z)e^{iz}dz$$

Integration using residues. Type 4.

$$I := \int_0^\infty \frac{R(x)}{x^\alpha}\,dx \qquad 0<\alpha<1$$

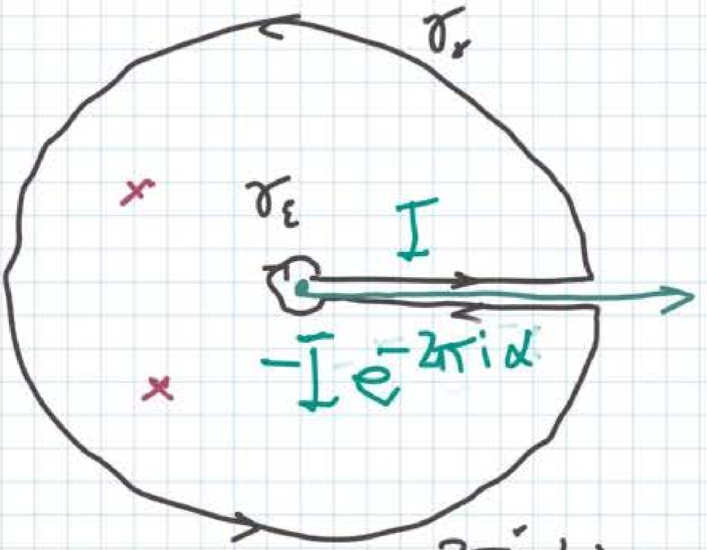
R – rational function vanishing a ∞, without poles in $\mathbb{R}_{\geq 0}$

Convergence: $\int_0^1 \frac{dx}{x^\alpha} = \frac{x^{1-\alpha}}{1-\alpha}\Big|_0^1 = \frac{1}{1-\alpha} < \infty$

$$\int_1^\infty \frac{dx}{x^{1+\alpha}} = \frac{x^{-\alpha}}{-\alpha}\Big|_1^\infty = \frac{1}{\alpha} < \infty$$

$$\int_0^{2\pi} \frac{R(re^{i\theta})}{r^\alpha e^{i\alpha\theta}}\, r i e^{i\theta}\, d\theta \to 0$$

bounded at ∞



$$\int_0^{2\pi} \frac{R(\varepsilon e^{i\theta})\,\varepsilon\, i e^{i\theta}\, d\theta}{\varepsilon^\alpha e^{i\alpha\theta}} \to 0 \text{ as } \varepsilon \to 0$$

$$I\left(1-e^{-2\pi i\alpha}\right) = 2\pi i \sum \operatorname{Res}_{z_k} \frac{R(z)\,dz}{z^\alpha}$$

Example: $I := \int_0^\infty \frac{dx}{x^\alpha(1+x)^2}, \quad 0<\alpha<1$

$$I\left(1-e^{-2\pi i\alpha}\right) = 2\pi i \operatorname{Res}_{z=-1} \frac{dz}{z^\alpha(1+z)^2}$$

$$= 2\pi i \left(\frac{d}{dz} z^{-\alpha}\right)\Big|_{z=e^{\pi i}} = 2\pi i\,(+\alpha)\, e^{-\pi i\alpha \cancel{-\pi i}}$$

$$I = \pi\alpha\,\frac{2ie^{-\pi i\alpha}}{1-e^{-2\pi i\alpha}} = \frac{\pi\alpha}{\sin\pi\alpha}$$

Remark: $\int_0^\infty R(x)\,dx = \lim_{\alpha\to 0^+} \int_0^\infty \frac{R(x)\,dx}{x^\alpha}$

$xR(x) \to 0$ as $x \to \infty$ no poles in $\mathbb{R}_{\geq 0}$

Example: $\int_0^\infty \frac{dx}{(1+x)^2} = -\frac{1}{1+x}\Big|_0^\infty = 1$

$$\lim_{\alpha\to 0^+} \frac{\pi\alpha}{\sin\pi\alpha} = 1$$

Example: $I := \int_0^\infty \frac{dx}{1+x^5}$

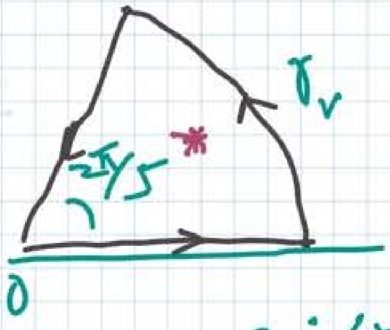
1st method $\quad x = y^{1/5}$

$$I = \frac{1}{5}\int_0^\infty \frac{dy}{y^{4/5}(1+y)}, \quad \alpha = 4/5$$

$$I\left(1 - e^{-2\pi i \cdot 4/5}\right) = \frac{2\pi i}{5} \operatorname*{Res}_{z=-1} \frac{dz}{z^{4/5}(1+z)}$$

$$I = \frac{\pi/5}{\sin \cancel{4}\pi/5} \qquad = \frac{2\pi i}{5} e^{-4\pi i/5}$$

2nd method



$$\oint \frac{dz}{1+z^5} = \int_0^r \frac{dx}{1+x^5} + \int_r^0 \frac{dx\, e^{\pi i/5}}{1+x^5} + \int_{\gamma_r} \frac{dz}{1+z^5}$$

$\int_{\gamma_r} \to 0$ as $r \to \infty$

$$I\left(1 - e^{2\pi i/5}\right) = 2\pi i \operatorname*{Res}_{z = e^{\pi i/5}} \frac{dz}{1+z^5}$$

Digression: $\frac{P(z)}{Q(z)} = \sum \frac{A_k}{z - z_k}$

$\deg P < \deg Q$ $\qquad$ simple roots

$$A_k = \operatorname*{Res}_{z_k} \frac{P(z)\,dz}{Q(z)} = \lim_{z \to z_k} \frac{P(z)(z - z_k)}{Q(z)} = \frac{P(z_k)}{Q'(z_k)}$$

$$= \frac{2\pi i \,(-z)}{5\, \cancel{z^4}}\bigg|_{z = e^{\pi i/5}} = -\frac{2\pi i}{5} e^{\pi i/5} = I\left(1 - e^{2\pi i/5}\right)$$

3rd Method $\quad \lim_{\alpha \to 0^+} \int_0^\infty \frac{dx}{x^\alpha (1+x^5)}$

$$\left(1 - e^{-2\pi i \alpha}\right) I(\alpha) = 2\pi i \sum_{z_k^5 = -1} \frac{-z_k^{-\alpha+1}}{5 \cancel{z_k^4}}$$

l'Hospital: $I(0) = \cancel{+}\, \frac{1}{5} \sum z_k \underbrace{\log z_k}_{= \pi i (2k+1)/5}$ $\quad = \frac{\pi/5}{\sin \pi/5}$

$k = 0, 1, 2, 3, 4$

4th method

$$\int \frac{dz}{1+z^5} = \sum \frac{1\,(-z_k)}{5 \cancel{z_k^4}} \int \frac{dz}{z - z_k} = -\frac{1}{5} \sum z_k \log(z - z_k)\Big|_0^\infty$$

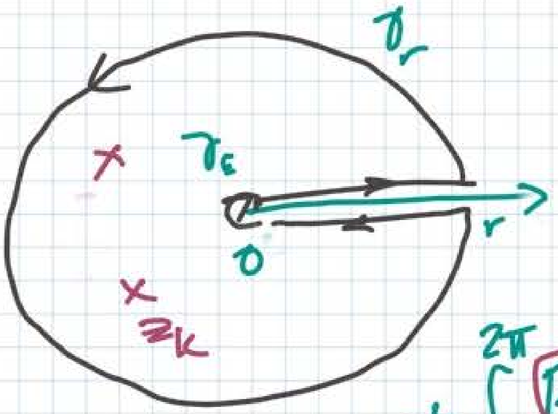
Integration Using Residues. Type 5

$$I := \int_0^\infty R(x)\log x\,dx$$

R — rational, no poles on $\mathbb{R}_{\geq 0}$, $R(x)x \to 0$ as $x \to \infty$

Convergence: $\int_0^1 \log x\,dx = (x\log x - x)\Big|_0^1 = -1$

$$\int_1^\infty \frac{\log x}{x^2}\,dx \underset{x=1/y}{=} -\int_0^1 \log y\,dy = 1$$

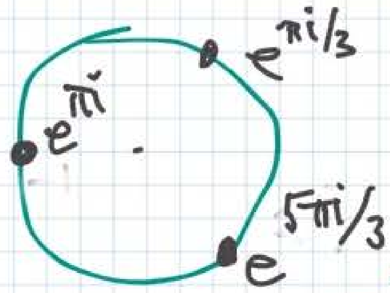


$$\oint R(z)(\log z)^2\,dz$$
$$= \int_\varepsilon^r R(x)(\log x)^2\,dx$$
$$- \int_\varepsilon^r R(x)(\log x + 2\pi i)^2\,dx$$
$$+ \int_0^{2\pi} R(re^{i\theta})(\log r + i\theta)^2\, r i e^{i\theta}\,d\theta \to 0 \text{ as } r \to \infty$$
$$\cancel{+} \int_0^{2\pi} R(\varepsilon e^{i\theta})(\log \varepsilon + i\theta)^2\, \varepsilon i e^{i\theta}\,d\theta \to 0 \quad (\varepsilon \to 0)$$

$$\xrightarrow[\varepsilon \to 0]{r \to \infty} \; -\overset{2}{\cancel{4\pi i}}\, I - (2\pi i)^{\cancel{2}} \int_0^\infty R(x)\,dx$$
$$= \cancel{2\pi i} \sum \mathrm{Res}_{z_k} (\log z)^2 R(z)\,dz$$

Example: $I = \int_0^\infty \frac{\log x}{1+x^3}\,dx$

$$-2I - 2\pi i \int_0^\infty \frac{dx}{1+x^3} = \sum_{z_k^3 = -1} \mathrm{Res}_{z_k} \frac{(\log z)^2\,dz}{1+z^3}$$



$$\frac{\left(\frac{\pi i}{3}\right)^2}{3e^{2\pi i/3}} + \frac{(\pi i)^2}{3} + \frac{\left(\frac{5\pi i}{3}\right)^2}{3e^{10\pi i/3}} =$$
$$-\frac{\pi^2}{27}\left(-\frac{1}{2} - i\frac{\sqrt3}{2}\right) - \frac{\pi^2}{3} - \frac{25\pi^2}{27}\left(-\frac{1}{2} + i\frac{\sqrt3}{2}\right)$$
$$= \frac{4}{27}\pi^2 - i\frac{4}{9}\sqrt3\,\pi^2 \qquad I = -\frac{2}{27}\pi^2, \quad \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt3}$$

Check: $J := \int_0^\infty \frac{dx}{1+x^3} = \frac{1}{3}\int_0^\infty y^{-2/3}\frac{dy}{1+y}$

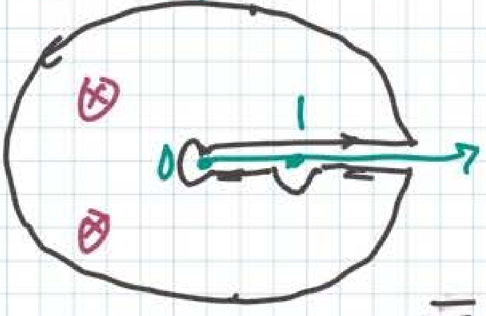
$$J\left(1 - e^{-4\pi i/3}\right) = \frac{2\pi i}{3}\,\mathrm{Res}_{y=-1}\, y^{-2/3}\frac{dy}{1+y} = \frac{2\pi i}{3} e^{-2\pi i/3}$$

$$J = \frac{\pi/3}{\sin \pi/3} = \frac{\pi}{3}\frac{2}{\sqrt3} \;!$$

Integration using residues, Type 5 (cont.d)

$$I = \int_0^\infty R(x)\log x\,dx$$

as before but has 1st order pole at $x=1$.

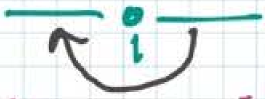


$$\lim_{\substack{\varepsilon,\varepsilon'\to 0\\ r\to\infty}} \oint R(z)(\log z)^2\,dz$$

$$= -\cancel{4\pi i}\, I - (2\pi i)^2 \int_0^\infty R(x)\,dx$$

real

"principal value"



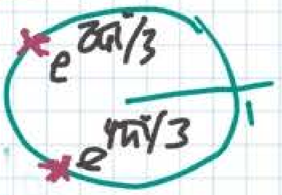
$$\int \frac{dz}{z-1} = -\pi i$$

$$-(\cancel{2}\pi i)^2\,\pi i\, \underbrace{\mathrm{Res}_{z=1} R(z)\,dz}_{\text{real}}$$

$$= 2\pi i \sum \mathrm{Res}_{\otimes} R(z)(\log z)^2\,dz$$

$$\boxed{I = \pi^2 \mathrm{Res}_1 R\,dz - \frac{1}{2}\mathrm{Re}\sum \mathrm{Res}_{\otimes} R(z)(\log z)^2\,dz}$$

Example: $I = \int_0^\infty \frac{\log x}{x^3-1}\,dx$



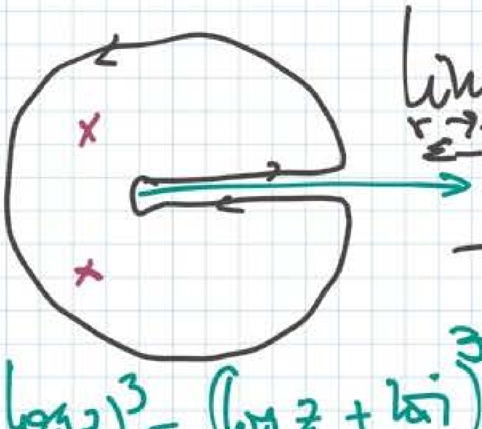
$$= \pi^2 \mathrm{Res}_1 \frac{dz}{z^3-1} - \frac{1}{2}\mathrm{Re}\,\mathrm{Res}_{\otimes} \frac{(\log z)^2\,dz}{z^3-1}$$

$$= \frac{\pi^2}{3} - \frac{1}{2}\mathrm{Re}\left[\frac{(2\pi i/3)^2}{3(e^{2\pi i/3})^2} + \frac{(4\pi i/3)^2}{3(e^{4\pi i/3})^2}\right]$$

$$= \frac{\pi^2}{3} + \mathrm{Re}\left[\frac{2\pi^2}{27}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) + \frac{8\pi^2}{27}\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi^2\left(\frac{1}{3}-\frac{5}{27}\right) = \frac{4}{27}\pi^2$$

Remark: $J := \int_0^\infty R(x)(\log x)^2\,dx = ??$



$$\lim_{\substack{r\to\infty\\ \varepsilon\to 0}} \oint R(z)(\log z)^3\,dz =$$

$$-6\pi i J - 3(2\pi i)^2 \int_0^\infty R(x)\log x\,dx$$

$$-(2\pi i)^3 \int_0^\infty R(x)\,dx$$

$$(\log z)^3 - (\log z + 2\pi i)^3$$

and so on...

Harmonic functions

$$\Delta g = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\in C^2$, i.e. 2 times continuously differentiable.

$$\Delta = \underbrace{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)}_{2\frac{\partial}{\partial z}}\underbrace{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{2\frac{\partial}{\partial \bar z}} = 4\frac{\partial^2}{\partial z \partial \bar z}$$

Remark: $i\frac{\partial^2}{\partial x \partial y} = i\frac{\partial^2}{\partial y \partial x}$ since $g \in C^2$

Corollary: Holomorphic functions are harmonic.

Proof: $\frac{\partial f}{\partial \bar z} = 0 \Rightarrow \frac{\partial^2 f}{\partial z \partial \bar z} = 0$

Corollary: Re f, Im f are harmonic.

Theorem: g - real harmonic $\Rightarrow g = \mathrm{Re} f$ (locally!) where f is holomorphic, unique up to an imaginary constant

Proof: $2\frac{\partial g}{\partial z}$ is holom. $\Rightarrow 2\frac{\partial g}{\partial z}\,dz \underset{\text{locally}}{=} d\underset{\text{holom.}}{f(z)}$

$\Rightarrow d\,\overline{f(z)} = 2\overline{\frac{\partial g}{\partial z}}\,\overline{dz}$ (real)

$\Rightarrow \frac{1}{2}d\left[f(z) + \overline{f(z)}\right] = dg \Rightarrow g = \mathrm{Re} f + \text{const}$

Uniqueness: $f = u + iv$, $u \equiv 0 \underset{CR}{\Rightarrow} v_x = v_y \equiv 0$.

Counter-example: $\log z = \log|z| + i\arg(z)$

undefined in $\mathbb{C} \setminus 0$ — harmonic in $\mathbb{C} \setminus 0$

Proposition: D - simply connected

$\Rightarrow$ g harmonic in D, $= \mathrm{Re}(f$ holom. in $D)$

Proof: $2\frac{\partial g}{\partial z}\,dz = df$ globally.

Corollary: Harmonic funct. possess the M.V.P.

$\Rightarrow$ Real harmonic funct. satisfy the max. ~~modulus~~ principle ($g + \text{const} > 0$ on compact D)

Analyticity of harmonic functions

$$f(z) = \sum_{n\geq 0} a_n (x+iy)^n = \sum_{p,q\geq 0} a_{p+q} \binom{p+q}{q} x^p (iy)^q$$

If $\sum |a_n| \rho^n < \infty \Rightarrow$ converges normally in $|x|, |y| \leq \rho/2$

$$\sum_{p,q\geq 0} |a_{p+q}| \binom{p+q}{q} |x|^p |y|^q \leq \sum_{n\geq 0} |a_n| \rho^n < \infty$$

Corollary: f, $\mathrm{Re} f$, $\mathrm{Im} f$ – analytic functions in 2 variables.

Corollary: Harmonic functions are analytic. (in particular, C^∞).

Theorem: If g is real harmonic in $x^2+y^2 < \rho^2$ then $g(x,y) = \underline{\mathrm{Re}} \left[2g\left(\frac{z}{2}, \frac{z}{2i}\right) - g(0,0) \right]$

Example: $g(x,y) = \dfrac{\sin x \cos x}{\cos^2 x + \sinh^2 y}$, $g(0,0)=0$

$$2g\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{2 \sin\frac{z}{2} \cos\frac{z}{2}}{\cos^2\frac{z}{2} + \cancel{\sinh^2 \frac{z}{2i}}} = \frac{\sin z}{\cos z} = \tan z$$

$$- \sin^2 z/2$$

Exercise: $g(x,y) = \mathrm{Re} \tan(x+iy)$

Proof of Theorem: $g(x,y) = \mathrm{Re} \sum a_n (x+iy)^n$

$$\Rightarrow 2g(x,y) = \sum a_n (x+iy)^n + \sum \bar{a}_n (x-iy)^n$$

$$\Rightarrow 2g\left(\frac{z}{2}, \frac{z}{2i}\right) = \sum a_n \left(\frac{z}{2}+\frac{z}{2}\right)^n + \bar{a}_0$$

$$2g(0,0) = a_0 + \bar{a}_0$$

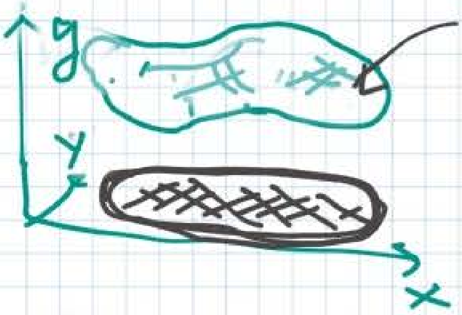
$$\Rightarrow \underbrace{2g\left(\frac{z}{2}, \frac{z}{2i}\right) - g(0,0)}_{} = \sum_{n\geq 0} a_n z^n + \underbrace{\tfrac{1}{2}(\bar{a}_0 - a_0)}_{\text{imaginary!}}$$

Remark: For any analytic g in 2 variables, it is defined, but has real part $g(x,y)$ if (Theorem) and only if g is harmonic (because Re (holom.) is harmonic)

Dirichlet's Problem

Find a cont. function in $\overline{D}$ which is harmonic in D and equal to a given continuous function on ∂D.

We'll solve this problem for a disk:



$$\text{Area} = \iint_D \sqrt{1+g_x^2+g_y^2}\,dx\,dy$$

$$= \text{Area}(D) + \underbrace{\tfrac{1}{2}\iint_D (g_x^2+g_y^2)\,dx\,dy}_{\min \Rightarrow \Delta g = 0} + \dots$$

Some (complex) solutions:

f is holom. in a nbhd of $\overline{D}$

$$f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(\xi)\,d\xi}{\xi - z}, \quad z \in D$$

↑ holom. $\Rightarrow$ harmonic, $= f|_{\partial D}$ on ∂D.

Some real solutions for D = disk:

$f(z) = \sum a_n z^n$, $a_0 \in \mathbb{R}$, $g = \operatorname{Re} f$

$$g\Big|_{|\xi|=r} = \underbrace{a_0 + \tfrac{1}{2}\sum_{n>0} a_n \xi^n + \tfrac{1}{2}\sum_{n>0} \bar{a}_n \xi^{-n}}_{\text{Fourier series}}, \quad \xi = e^{i\theta} r$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\,d\theta$$

$$r^n a_n = \frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\,e^{-in\theta}\,d\theta, \quad n>0$$

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\underbrace{\Big[1 + 2\sum_{n\ge 1}\Big(\frac{z}{re^{i\theta}}\Big)^n\Big]}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\,\frac{1+z/re^{i\theta}}{1-z/re^{i\theta}}\,d\theta \qquad \frac{2z/re^{i\theta}}{1-z/re^{i\theta}} + \frac{1-z/re^{i\theta}}{1-z/re^{i\theta}}$$

$$g(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{g(r\cos\theta, r\sin\theta)}_{\substack{\text{can be replaced}\\ \text{with any cont.}\\ \text{function of } \theta}}\ \underbrace{\operatorname{Re}\Big[\frac{re^{i\theta}+z}{re^{i\theta}-z}\Big]}_{\text{Poisson's kernel}}\,d\theta$$

$z = x+iy$

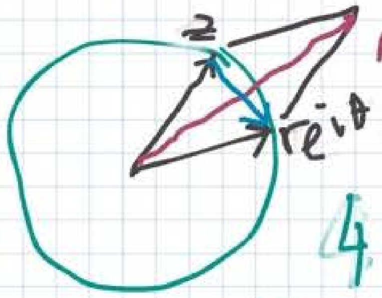
Properties of Poisson's kernel

$$\mathrm{Re}\,\frac{re^{i\theta}+z}{re^{i\theta}-z} = \frac{r^2-|z|^2}{|re^{i\theta}-z|^2} \quad (r\text{- fixed})$$

1° for $z \neq re^{i\theta}$, ~~holom~~ harmonic in z (continuous in (z,θ))

2° Positive for $|z|<r$

3° Vanishes for $|z|=r$, $z\neq re^{i\theta}$:



$\dfrac{re^{i\theta}+z}{re^{i\theta}-z}$ – purely imaginary

$$4° \quad 1 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{re^{i\theta}+z}{re^{i\theta}-z}\,d\theta$$

for any fixed z with $|z|<r$.

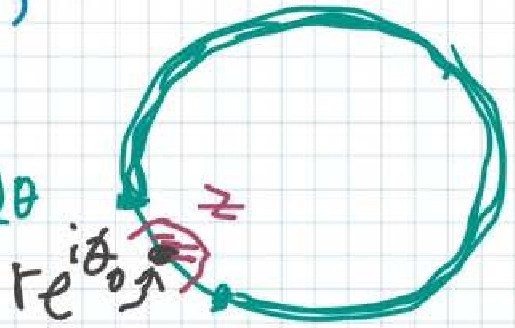
Conclusion: As z approaches $re^{i\theta}$, Poisson's kernel tends to Dirac's δ-function on the circle, concentrated at $re^{i\theta}$.

Theorem: Given a continuous 2π-periodic φ,

$$g(z) := \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

is harmonic in $|z|<r$, and

$$\lim_{z\to re^{i\theta}} g(z) = \varphi(\theta).$$



Proof. $\varphi(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta_0)[\ldots]\,d\theta$

$$g(z)-\varphi(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi}(\varphi(\theta)-\varphi(\theta_0))\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} = \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} + \frac{1}{2\pi}\int_{\theta_0+\delta}^{\theta_0+2\pi-\delta} =: A+B$$

$\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t. $|\varphi(\theta)-\varphi(\theta_0)|<\frac{\varepsilon}{2}$ if $|\theta-\theta_0|<\delta$.

$|A| \le \varepsilon/2$ ($2° + 4°$), $|B|<\varepsilon/2$ when z is close to $re^{i\theta_0}$

$\lim_{z\to re^{i\theta_0}} B = 0$ ($3°$ + Continuity in z,θ on $\times\,[\theta_0+\delta,\ \theta_0+2\pi-\delta]$)

Spaces of functions $\mathcal{H}(D) \subset C(D)$

holomorphic, continuous, open set in $\mathbb{C}$

$f_n \to f$ uniformly on compact subsets

if $\max_{z\in K} |f_n(z) - f(z)| \to 0$ as $n\to\infty$ for every compact $K \subset D$

Remarks: 1° $f_n \in C(D) \Rightarrow f \in C(D)$

2° It suffices to check uniform convergence on closed disks in D

[Every compact $K \subset D$ is covered by the interiors of finitely many such disks, $\max_K \le \max_i \max_{K_i}$]

Theorem 1. $f_n \in \mathcal{H}(D) \Rightarrow f \in \mathcal{H}(D)$

Proof 1: $\int_{\triangle} f\,dz = \lim_{n\to\infty} \int_{\triangle} f_n\,dz = 0$

$\Rightarrow f\,dz\ (+0\,d\bar z) = dg \Rightarrow g \in \mathcal{H}(D),\ f = g'$.

Proof 2: $f_n(z) = \frac{1}{2\pi i}\oint_{|t|=r} \frac{f_n(t)\,dt}{t-z} \to \frac{1}{2\pi i}\oint_{|t|=r} \frac{f(t)\,dt}{t-z} = f(z)$

pointwise for each z, $|z|<r$ $\Rightarrow$ holom.

Example: $\frac{\sin nx}{\sqrt n} \to 0$, but $\sqrt n \cos nx \not\to 0$

Theorem 2. In $\mathcal{H}(D)$, $f_n \to f \Rightarrow f_n' \to f'$

Proof: $|f_n'(z) - f'(z)| = \left|\frac{1}{2\pi i}\int_{|t|=r} \frac{f_n(t)-f(t)}{(t-z)^2}\,dt\right| \le$ ($|z| \le r/2$)

$\max_{|t|=r} |f_n(t) - f(t)|\, r / (r/2)^2 \to 0$

uniformly in

$\sum f_n$ converges normally on compact subsets in D

if $\sum_n \max_{z\in K} |f_n(z)| < \infty$ for all compact $K \subset D$.

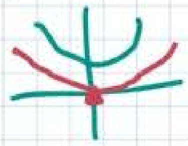
$\sum f_n$ conv. normally $\Rightarrow S_n := \sum_{i=1}^{n} f_i$ conv. uniformly

Corollary: $f_n \in \mathcal{H}(D) \Rightarrow \sum f_n \in \mathcal{H}(D) \Rightarrow \sum f_n' = (\sum f_n)'$

One (counter-intuitive) application

If non-vanishing holom. functions f_n converge (uniformly on compact subsets) in a connected open set D to a non-zero function f, then f is also non-vanishing.

In real analysis, $x^2+\frac{1}{n} \to x^2$



Proof: If z_0 is an isolated zero of f, then $\oint_{|z-z_0|=\varepsilon} \frac{f'}{f}\,dz = 2\pi i \times (\text{multiplicity}) \neq 0$

But $f_n \to f$, $f_n' \to f' \Rightarrow \frac{f_n'}{f_n} \to \frac{f'}{f}$ on $|z-z_0|=\varepsilon$

where $f \neq 0$: $f_n \to f,\ g_n \to g \Rightarrow$

$f_n g = [f + (f_n - f)]g \to fg$ bounded in K

$f_n g_n = [f + (f_n - f)][g + (g_n - g)] \to fg$,

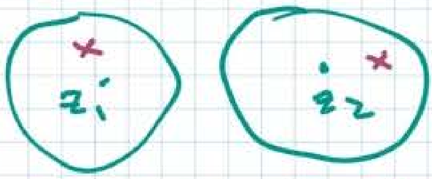
$\frac{1}{f} - \frac{1}{f_n} = \frac{f_n - f}{f f_n} \to 0$ if $f \neq 0$ $\left(|f_n| \geq \frac{\min|f|_K}{2}\right)$

$\Rightarrow \oint \frac{f'}{f}\,dz = \lim \oint \frac{f_n'}{f_n}\,dz = 0$ - contradiction!

Corollary: If a sequence $f_n : D \to \mathbb{C}$ of simple (= injective) holom. functions converge to f (uniformly on compact subsets) then f is also injective — unless constant.

Proof: Suppose not: $f(z_1) = f(z_2) =: a$

Then $f - a$ has an isolated zero in $|z - z_1| < \varepsilon_1$ and $|z - z_2| < \varepsilon_2$ (disjoint disks).



Then for (n large enough) $f_n - a$ must have a zero in each disk

— contradiction with the injectivity of f_n.

Remark: $\frac{1}{2\pi i}\oint_{|z-z_0|=\varepsilon} \frac{f'(z)\,dz}{f(z)-a} = \#$ solutions of $f(z)=a$ for $|z-z_0|<\varepsilon$

$a \approx f(z_0)$

$= 1$ if only if $f'(z_0) \neq 0$

Alternatively $f(z) - f(z_0) = (z - z_0)^k g = [(z - z_0)\, g^{1/k}]^k$ - k-fold

So, f (non-const) $\Leftarrow f_n$ - locally injective $\Rightarrow f$ - locally injective $\left(f_n' \neq 0 \Rightarrow f' \neq 0\right)$

$\mathcal{H}(D) \subset \mathcal{C}(D)$ as metric spaces

1° Cover D by countably many compact subsets $K_i \subset D$, $\bigcup_{i=1}^{\infty} K_i = D$.

E.g. take disks $|z - z_i| \le r_i$ fitting D with rational r_i, $\operatorname{Re} z_i$, $\operatorname{Im} z_i$.

2° $f_n \to f$ uniformly on compacts in D

$\Leftrightarrow$ $\|f_n - f\|_i := \max_{z \in K_i} |f_n(z) - f(z)| \longrightarrow 0$ as $n \to \infty$ for each $i = 1, 2, \dots$

$\|\cdot\|_i$ – "seminorm"

$\|f\| \ge 0$, $\|f+g\| \le \|f\| + \|g\|$, $\|\lambda f\| = |\lambda| \|f\|$

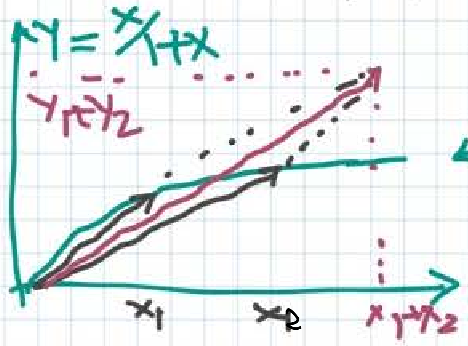
could be 0 for non-zero f

So, $\mathcal{C}(D)$ - a countably normed space

All $\|f\|_i = 0 \Rightarrow f = 0$

3° Theorem: A countably normed space is metric (another name: Frechet's space)

$$d(f,g) := \sum_{i=1}^{\infty} \frac{1}{2^i} \underbrace{\|f-g\|_i / 1 + \|f-g\|_i}_{\text{semi-metric} < 1}$$



← increasing, convex down

$x_n \to 0 \Leftrightarrow Y_n \to 0$ ($Y(0) = 0$, continuous)

$x \le x_1 + x_2 \Rightarrow Y \le Y_1 + Y_2$

(i) $0 \le d(f,g) < 1$, $d(f,g) = 0 \Leftrightarrow f = g$

(ii) $d(f,g) = d(g,f)$, (iii) $d(f,h) \le d(f,g) + d(g,h)$

d is a metric

$f_n \to f \Leftrightarrow \|f_n - f\|_i / 1 + \|f_n - f\|_i \to 0$ for each i

$\Rightarrow \forall k.\ d(f_n, f) < \frac{1}{2^{k-1}} = \frac{1}{2^k} + \frac{1}{2^k}$ for n large enough

$\frac{1}{2^k} > \sum_{i > k} 2^{-i}$

$\Rightarrow d(f_n, f) \to 0$ as $n \to \infty$

$\Leftarrow$ $\|f_n - f\|_i / 1 + \|f_n - f\|_i \le 2^i d(f_n, f)$

Properties of metric space $\mathcal{H}(D) \subset (\mathcal{C}(D), d)$

- $\mathcal{C}(D)$ is a complete metric space
- $\mathcal{H}(D)$ is closed in $\mathcal{C}(D)$ hence complete!
- $\mathcal{H}(D) \to \mathcal{H}(D)$: $f \mapsto f'$ is continuous

Series of meromorphic functions

Example: $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = ?$

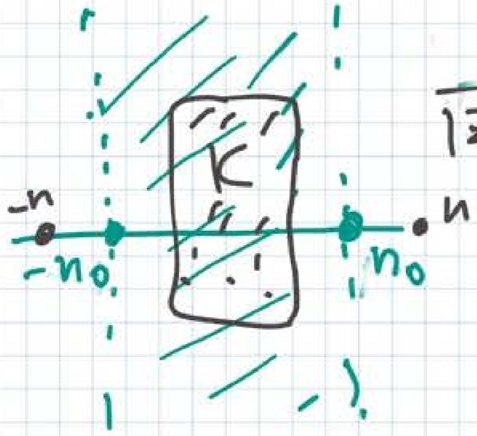
Def. $\sum f_n$ (meromorphic in D) converges uniformly/normally on compact subsets in D if for every compact $K\subset D$ there exists N such that $\sum_{n\geq N} f_n$ converges uniformly/normally on K.

Taking $K=\overline{U}\subset D$ for an open U we find

$$\sum f_n = \underbrace{\sum_{n<N} f_n}_{\text{merom.}} + \underbrace{\sum_{n\geq N} f_n}_{\text{holom}}$$

– meromorphic with finitely many poles in U

$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ converges normally on compact subsets in $\mathbb{C}$.



$$\frac{1}{|z-n|^2} \leq \frac{1}{(|n|-n_0)^2} \quad \text{for } |n|>n_0 \text{ and } |\mathrm{Re}\, z|\leq n_0$$

and $\sum_{|n|>n_0} \frac{1}{(|n|-n_0)^2} < \infty$
$= 2\sum_{k=1}^{\infty}\frac{1}{k^2}$

Theorem: $$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2$$

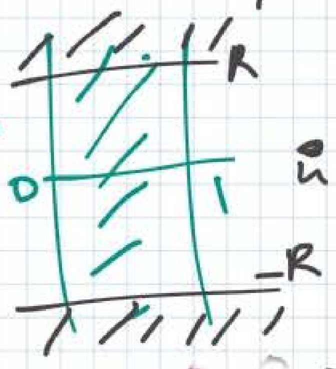
1° Both are even and 1-periodic.

2° Both have 2-nd order poles at $z=n\in\mathbb{Z}$ with principal part $\frac{1}{(z-n)^2}$

periodicity + parity: $\frac{1}{z^2}$ + holom / $\left(\frac{\pi}{\pi z+\dots}\right)^2 = \frac{1}{z^2}+\dots$

3° Both tend to 0 as $\mathrm{Im}\, z\to\infty$ uniformly in $\mathrm{Re}\, z$

Each $\frac{1}{(z-n)^2}\to 0$ uniformly as $|\mathrm{Im}\, z|\to\infty$



$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi (y) \to \infty$

4° $\sum \frac{1}{(z-n)^2} - \left(\frac{\pi}{\sin \pi z}\right)^2$ – holom in $\mathbb{C}$, $\to 0$ as ∞ $\Rightarrow\ \equiv 0$

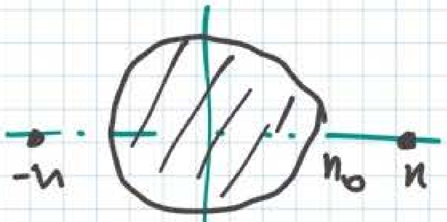
Series of meromorphic functions (cont'd) 25.1

Last time: $\left(\frac{\pi}{\sin \pi z}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \left[\frac{1}{z-n}+\frac{1}{n}\right] + \frac{1}{z} = \frac{1}{z} + \sum_{n\neq 0} \frac{z}{n(z-n)}$$

$$|z| \le n_0 \Rightarrow \left|\frac{z}{n(z-n)}\right| \le \frac{n_0}{(|n|-n_0)^2}$$

for $|n| > n_0$



normal conv. on compact subsets. $\Leftarrow \sum_{n>n_0} \frac{1}{(n-n_0)^2} \left(=\pi^2/6\right) < \infty$

$$F(z) := \frac{1}{z} + \sum_{n\neq 0}\left[\frac{1}{z-n}+\frac{1}{n}\right]$$ - meromorphic in $\mathbb{C}$

- has 1-st order poles at $z=n$ with Res $=1$.
- periodic with period 1
- odd $\left(\frac{-z}{n(-z-n)} = -\frac{z}{-n(z+n)}\right)$

$$\frac{dF}{dz} = -\frac{1}{z^2} - \sum_{n\neq 0} \frac{1}{(z-n)^2} = -\left(\frac{\pi}{\sin \pi z}\right)^2$$

$$= \frac{d}{dz}\left(\frac{\pi}{\tan \pi z}\right) \qquad \boxed{\cot' = -\operatorname{cosec}^2}$$

$$\Rightarrow F(z) = \frac{\pi}{\tan \pi z} = \frac{1}{z} + \sum_{n\ge 1} \frac{2z}{z^2-n^2}$$

$$\frac{z}{n(z-n)} + \frac{z}{-n(z+n)} = \frac{\cancel{z^2}+\cancel{n}z - \cancel{z^2}+\cancel{n}z}{\cancel{n}(z^2-n^2)}$$

Corollary: $\pi z \cot \pi z - 1 =$

$$-2\sum_{n\ge 1} \frac{z^2}{n^2(1-z^2/n^2)} = -2\sum_{n\ge 1}\sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}}$$

$$= -2\sum_{k=1}^{\infty} \zeta(2k)\, z^{2k}$$

$\zeta(2k)$ via Bernoulli numbers (once again)

$$1-2\sum_{k=1}^{\infty}\zeta(2k)z^{2k} \underset{|z|<1}{=} \pi z\cot \pi z$$

$$= \frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}}\cdot \pi i z = \frac{\cancel{1}^{\,x\,1}+e^{-2\pi i z}\cancel{-1}}{1-e^{-2\pi i z}}\cdot \pi i z$$

$$= -\pi i z + \frac{2\pi i z}{1-e^{-2\pi i z}} = \cancel{-\pi i z}+1+\cancel{\frac{2\pi i z}{2}}$$

$$+\sum_{k=1}^{\infty} B_{2k}\frac{(2\pi i z)^{2k}}{(2k)!} \qquad \boxed{\frac{x}{1-e^{-x}} =: 1+\frac{x}{2}+\sum_{k>0} B_{2k}\frac{x^{2k}}{(2k)!}}$$

$$= 1+\sum_{k=1}^{\infty} B_{2k}\frac{(-1)^k(2\pi)^{2k}}{(2k)!}z^{2k} \Rightarrow$$

$$\boxed{\zeta(2k) = \pi^{2k}2^{2k-1}(-1)^{k-1}B_{2k}/(2k)!}$$

Check: $(k=1)$: $\pi^2 2(1/6)/2 = \pi^2/6$

$(k=2)$: $\pi^4\cdot 8\cdot(-1)(-1/30)/24 = \pi^4/90$

Another Example (from the book)

$$\left(\frac{\pi}{\sin \pi z}\right)^2\cos\pi z$$

2-periodic, even, 2nd order poles at $z=n$ with principal parts $\frac{(-1)^n}{(z-n)^2}$

$$\overset{?}{=} \sum_{n\in\mathbb{Z}}\frac{(-1)^n}{(z-n)^2}$$

Conv. uniformly on comp. subsets, $\to 0$ as $\operatorname{Im} z\to\infty$

$z=x+iy$

$$\frac{e^{\pi|y|}}{e^{2\pi|y|}} \xrightarrow[|y|\to\infty]{} 0$$

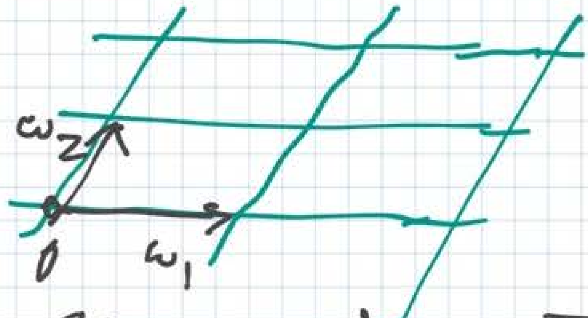
$$= -\frac{d}{dz}\left[\frac{1}{z}+\sum_{n\neq 0}\left[\frac{(-1)^n}{z-n}+\frac{(-1)^n}{n}\right]\right] = -\frac{d}{dz}\frac{\pi}{\sin \pi z}$$

odd

$$\Rightarrow \boxed{\frac{\pi}{\sin\pi z} = \frac{1}{z}+\sum_{n\geq 1}(-1)^n\frac{2z}{z^2-n^2}}$$

Doubly-periodic meromorphic functions

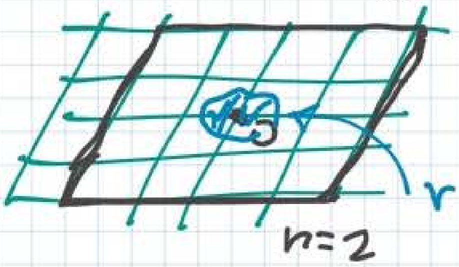
$\Omega = \{ m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z} \}$ period lattice



$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

- Converges normally on compact subsets

Lemma: $\displaystyle\sum_{\omega \in \Omega \setminus \{0\}} \frac{1}{|\omega|^3} < \infty$



$\max(|m_1|, |m_2|) = n$

$\Rightarrow$ $8n$ points. $|\omega| \geq n r$

$\Rightarrow \displaystyle\sum \leq \sum_{n=1}^{\infty} \frac{8n}{n^3 r^3} = \frac{8}{r^3}\zeta(2)$

$|z| \leq R$, $|\omega| \geq 2R$

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega z - z^2}{\omega^2 (z-\omega)^2} \right| = \frac{|z| \left| 2 - \frac{z}{\omega} \right|}{|\omega|^3 \left| 1 - \frac{z}{\omega} \right|^2}$$

normal convergence of $\displaystyle\sum_{|\omega| \geq 2R} (\ldots)$ on $|z| \leq R$. $\Leftarrow$ $\leq \dfrac{1}{|\omega|^3} \dfrac{2{,}5 R}{(0{,}5)^2}$

$\Rightarrow$ $\wp$ - meromorphic function with 2-nd order poles in Ω with principal part $\dfrac{1}{(z-\omega)^2}$ + (holom at ω) $= \wp(z)$,

$\wp(-z) = \wp(z)$ - even: $\begin{pmatrix} z \longleftrightarrow -z \\ \omega \longleftrightarrow -\omega \end{pmatrix}$

$\wp'(z) = \displaystyle\sum_{\omega \in \Omega} \frac{2}{(\omega - z)^3}$ - Ω-periodic, odd.

$\Rightarrow$ $\wp(z + \omega_i) - \wp(z) = \text{const}_i$; $(i = 1, 2)$

but: $\wp\left(-\frac{\omega_i}{2} + \omega_i\right) = \wp\left(-\frac{\omega_i}{2}\right) \Rightarrow \text{const}_i = 0$

$\Rightarrow$ $\wp$ - Ω-periodic.

Laurent expansions

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$= \frac{1}{z^2} + 0 \cdot z^0 + a_2 z^2 + a_4 z^4 + \ldots$$

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\left(1 + 2\frac{z}{\omega} + 3\frac{z^2}{\omega^2} + 4\frac{z^3}{\omega^3} + 5\frac{z^4}{\omega^4} + \ldots\right)$$

$$\Rightarrow \boxed{a_2 = 3 \sum_{\omega \neq 0} \frac{1}{\omega^4}}, \quad \boxed{a_4 = 5 \sum_{\omega \neq 0} \frac{1}{\omega^6}}$$

$$\wp' = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \ldots$$

$$(\wp')^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + O(z^2)$$

Ω-periodic, even

$$\wp^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + O(z^2)$$

$$\Rightarrow (\wp')^2 - 4\wp^3 = -\frac{20a_2}{z^2} - 28a_4 + O(z^2)$$

$$\Rightarrow \boxed{(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4 = 0}$$

Ω-periodic entire function vanishing at $z=0$.

Two interpretations

torus less one point

(i) $(\mathbb{C} \setminus \Omega)/\Omega = E \setminus \{pt\} \xrightarrow{(\wp, \wp')}$

$$\{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - 20a_2 x - 28a_4\}$$

elliptic curves

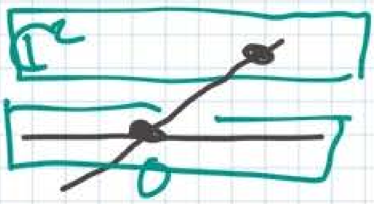
(ii) $m\ddot{x} = -\dfrac{dU(x)}{dx}$ (Newton eqn.)

$$\Rightarrow \frac{m\dot{x}^2}{2} + U(x) = \text{const} \quad (\text{energy conservation})$$

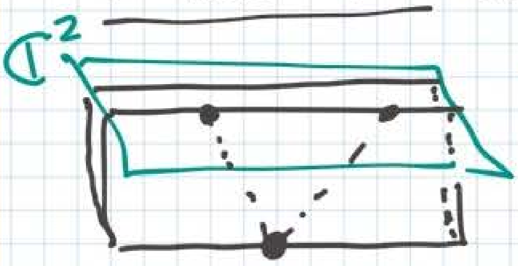
If U – a degree 3 polynomial, then solutions $t \mapsto x(t)$ are expressed via $z \mapsto \wp(z)$ and phase curves in $(x, \dot{x})$-plane are elliptic!

Plane Algebraic Geometry

$\mathbb{C}P^2$ = complex projective plane $:=$
$\{$ 1-dim subspaces in $\mathbb{C}^3 \}$
$= \mathbb{C}^2 \sqcup \underline{\mathbb{C}P^1}$ "points at infinity"

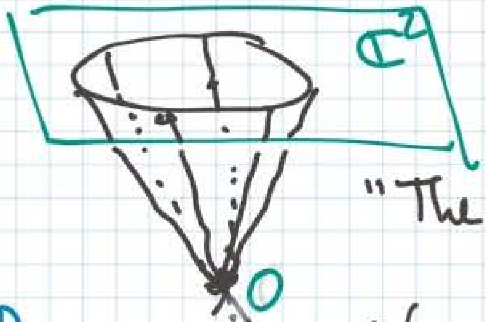


A line in $\mathbb{C}P^2$ = A 2-dim subspace in $\mathbb{C}^3$



"parallel lines in $\mathbb{C}^2$ intersect at one point at infinity"

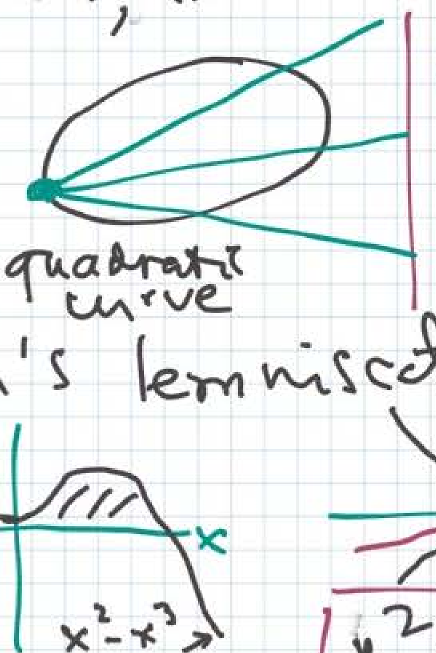
Algebraic curves in $\mathbb{C}P^2$: $F(z_1 : z_2 : z_3) = 0$
$=$ a conic surface in $\mathbb{C}^3$ homogeneous polynomial



"The conic" $z_1^2 + z_2^2 = z_3^2$

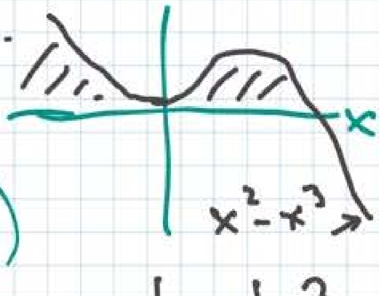
Proposition: Non-singular quadratic curves in $\mathbb{C}P^2$ are rational, i.e. $\simeq \mathbb{C}P^1$.

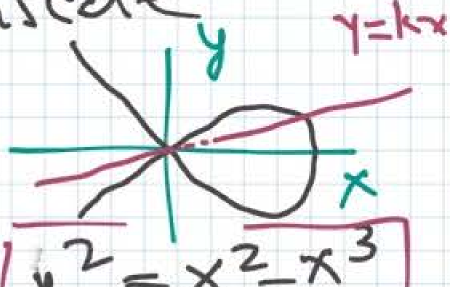
Proof: "stereographic projection"



Example: Bernoulli's lemniscate

$y = kx$





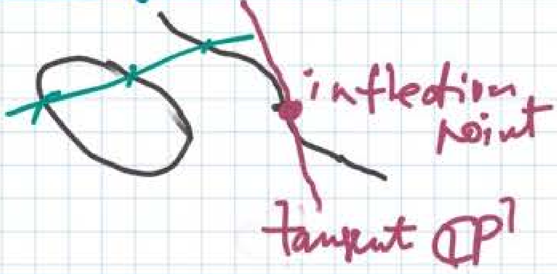
$\boxed{y^2 = x^2 - x^3}$

$k^2 \cancel{x^2} = \cancel{x^2}(1-x)$

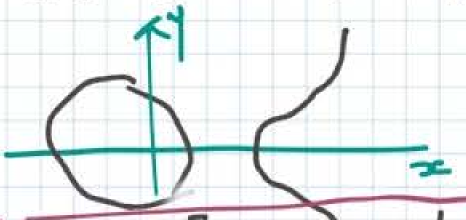
$x = 1 - k^2, \quad y = k - k^3$

$\mathbb{C}P^1$ with two points ($k = \pm 1$) identified

Elliptic curves: Nonsingular cubics in $\mathbb{C}P^2$.

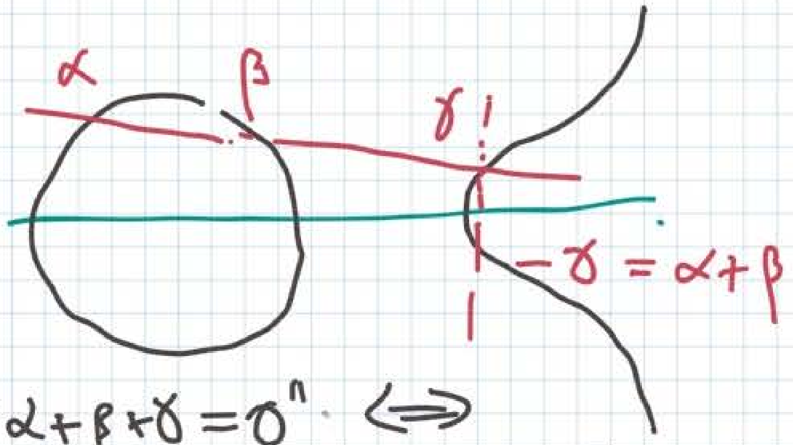


$\rightsquigarrow$



$\boxed{y^2 = x^3 + ax + b}$

A group structure on an elliptic curve



$y^2 = x^3 + ax + b$

$ZY^2 = X^3 + aZ^2X + bZ^3$

Point at ∞:

$Z = 0 \Rightarrow X^3 = 0$

$[X:Y:Z] = [0:1:0]$

"0" element, inflection point

"$\alpha+\beta+\gamma = 0$" $\Leftrightarrow$ "α, β, γ are colinear"

2nd order points: $g+g+0=0 \Leftrightarrow y=-y$

3rd order points $g+g+g=0 \Leftrightarrow$ inflection points

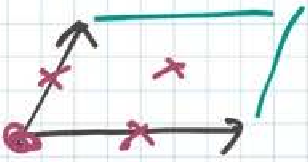
Theorem: $\mathbb{C}^1/\Omega \longrightarrow \mathbb{CP}^2$, $[\wp : \wp' : 1]$

- bijective parametrization of the elliptic curve
$$y^2 = 4x^3 - 20a_2 x - 28a_4.$$

Proof 1° $\wp = \frac{1}{z^2} + \dots$, $\wp' = -\frac{2}{z^3} + \dots$ point at infinity

$\Rightarrow [z^3\wp(z) : z^3\wp'(z) : z^3] \to [0:1:0]$ as $z \to 0$

2° At $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ mod Ω



$\wp'(z) = 0$ $\quad (\wp'(-z) = \wp'(z) = -\wp'(-z))$
periodicity, parity

$\Rightarrow \wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$ – roots of polynomial $4x^3 - 20a_2x - 28a_4$

simple zeroes of $\wp'$ $\Rightarrow$ distinct values of $\wp$ ($\Rightarrow$ simple roots).
(# zeros = # poles = 3 in each parallelogram).

3° For every x not a root of the polynomial $\wp = x$ has 2 solutions (# zeros = # poles = 2 in each parallelogram)
z & $-z$, with $\wp'(-z) = -\wp'(z)$

Proposition: The group structure is the same as in $\mathbb{C}/\Omega$

$$\frac{1}{2\pi i}\int_{\partial\Pi} z\,\frac{df}{f} = \sum_{f(z)=0} \alpha_i - \sum_{f(z)=\infty} \beta_i \in \Omega$$

Take $f = A\wp + B\wp' + C$

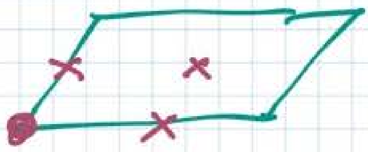
Corollary: There are 9 inflection points

Theorem: $\mathbb{C}/\Omega \xrightarrow[{[\wp : \wp' : 1]}]{} \mathbb{C}P^2$

is a group isomorphism of the complex torus to the non-singular cubical curve

$$y^2 = 4x^3 - 20a_2 x - 28a_4 .$$

Proof: ① $z_0 = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ $\quad z_0 \equiv -z_0 \mod \Omega$



$\wp'(z_0) = \wp'(-z_0) = -\wp'(z_0) \Rightarrow \wp'(z_0) = 0$

$\Rightarrow \wp(z)$ is a root of $4x^3 - 20a_2 x - 28a_4$

② $\#\text{zeroes} = \#\text{poles} = \begin{cases} 2 \text{ for } \wp(z) - \wp(z_0) \\ 3 \text{ for } \wp'(z) \end{cases}$

$\Rightarrow$ z_0 is the only (2nd order) zero of $\wp(z) - \wp(z_0)$

$\Rightarrow \wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$ are distinct

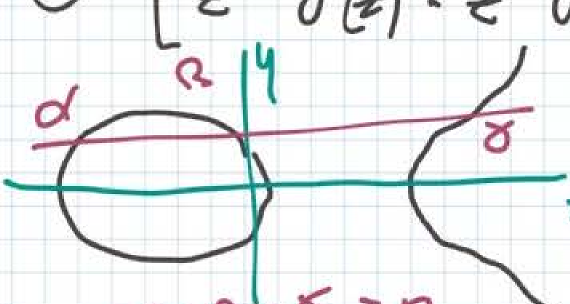
the only zeroes of $\wp'$

③ For any non-root x_0, $\wp(z) = x_0$ has 2 distinct (simple) solutions z_1, z_2

$\Rightarrow z_1 \equiv -z_2$ $(\wp(-z) = \wp(z)) \Rightarrow \wp'(z_1) = -\wp'(z_2)$

distinct!

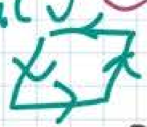
⓪ $[z^3\wp(z) : z^3\wp'(z) : z^3] \xrightarrow[z \to 0]{} [0 : 1 : 0]$ point at infinity



$\alpha + \beta + \gamma = 0$

④ Group isomorphism:

Recall: $\sum_{\text{zeroes}} a_i - \sum_{\text{poles}} b_j = \frac{1}{2\pi i} \int z \frac{df}{f} \in \Omega$



Apply to $f := A\wp(z) + B\wp'(z) + C$

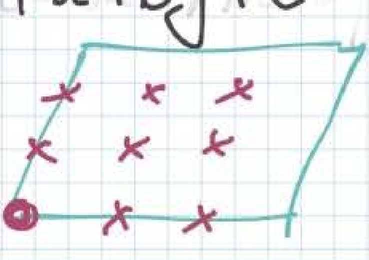
poles: $z=0$ of order 3 (if $B \neq 0$)

$\Rightarrow a_1 + a_2 + a_3 \equiv 0 \mod \Omega$

whenever $\begin{bmatrix} \wp(a_i) \\ \wp'(a_i) \end{bmatrix}$ lie on $Ax + By + C = 0$

Corollary:

There are 9 inflection points



Infinite products

$\prod_{n=1}^{\infty} f_n(z)$ converges normally on $K \subset D$ compact open

continuous

if $\| f_n - 1 \| \to 0$ as $n \to \infty$ and $\sum_{n > n_0}^{\infty} \| \log f_n \| < \infty$

$\| \cdot \| = \max_K | \cdot |$

$\Leftrightarrow \sum_{n=1}^{\infty} \| f_n - 1 \| < \infty$ $\quad \log(1+u) = u + O(u^2)$

$\frac{1}{2} \| u \| \leq \| \log(1+u) \| \leq 2 \| u \|$ for $|u|$ small enough

Theorem Suppose $\prod_{n=1}^{\infty} f_n(z)$ converges normally on compact subsets in D, where all f_n are holomorphic. Then $f := \prod f_n$ is holomorphic, $\sum \frac{f_n'}{f_n}$ conv. uniformly on comp. subsets to $\frac{f'}{f}$, zeroes of f are unions of zeros of f_n counting with multiplicities.

Proof. For $K = \overline{U}$, U open, $\exists n_0$: $\sum_{n > n_0} \log f_n$ conv. normally on U to holom. φ

$$f := \prod_{n=1}^{\infty} f_n = \prod_{n=1}^{n_0} f_n \, e^{\varphi}$$

in U — no zeros in U — holom

$$f^{-1}(0) = \bigcup_{n=1} f_n^{-1}(0)$$

$$\frac{f'}{f} = \sum_{n=1}^{n_0} \frac{f_n'}{f_n} + \varphi' \qquad \varphi' = \sum_{n > n_0} f_n'/f_n$$

Example: $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

① Convergence: $\sum_{n>0} \frac{z^2}{n^2}$ conv. normally on compact subsets

② Termwise logarithmic differentiation:

$$\frac{f'}{f} = \frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2} = \frac{\pi}{\tan \pi z} = \frac{(\sin \pi z)'}{\sin \pi z}$$

③ Normalization:

$$\lim_{z \to 0} \frac{\sin \pi z}{\pi z} = 1 = \lim_{z \to 0} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

The Gamma-function

$$\Gamma(z) := \int_0^\infty e^{-t} t^z \frac{dt}{t} \overset{\mathrm{Re}\, z>0}{=} \frac{1}{z}\Gamma(z+1)$$

$$+\ \Gamma(1) = 1 \Rightarrow \boxed{\Gamma(n+1) = n!}$$

Convergence $\int_0^\infty := \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \int_\varepsilon^M$

$$\left|\int_0^\varepsilon \cancel{e^{-t}} \dots\right| \le \frac{|t|^{\mathrm{Re}\, z}}{|z|}\Bigg|_0^\varepsilon = \frac{\varepsilon^{\mathrm{Re}\, z>0}}{|z|}$$

$$\left|\int_M^\infty (e^{-t/2})^2 \dots\right| \le e^{-M/2}\, 2^{\mathrm{Re}\, z}\, \Gamma(\mathrm{Re}\, z) \le \Gamma(\mathrm{Re}\, z)!$$

uniform on $0 < a \le \mathrm{Re}\, z \le b < \infty$

$\Rightarrow \Gamma$ is holomorphic in $\mathrm{Re}\, z > 0$

Analytical Continuation

$$e^{-t} = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$$

$$\Gamma(z) = \lim_{n\to\infty} \underbrace{\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt}$$

integration by part $\overset{}{=} \dfrac{n(n-1)\dots 1}{n\cdot n \dots n}\, \dfrac{n^{z+n}}{z(z+1)\dots(z+n)}$

$$= \lim_{n\to\infty}\left(\frac{n^z\, n!}{z(z+1)\dots(z+n)}\right) =: \frac{1}{g_n(z)}$$

$$f_n(z) := \frac{g_n(z)}{g_{n-1}(z)} = \left(\frac{n-1}{n}\right)^z \frac{z+n}{n} = \left(1+\frac{z}{n}\right)\left(1-\frac{1}{n}\right)^z$$

$$\frac{1}{\Gamma(z)} = \underbrace{g_1(z)\prod_{n\ge 2} f_n(z)} \qquad g_1 = z(z+1)$$

converges normally on compact subsets of $\mathbb{C}$

$$\log f_n = \left(\cancel{\frac{z}{n}} - \frac{z^2}{2n^2} + \frac{z^3}{3n^3} - \dots\right) - \left(\cancel{\frac{z}{n}} + \frac{z}{2n^2} + \frac{z}{3n^3} + \dots\right)$$

For $|z| \le r\ (>1)$ $\quad |\log f_n| \le \dfrac{2r^2}{n^2}$ for n large enough

Thus: $\dfrac{1}{\Gamma}$ is entire with simple zeros $0, -1, -2, \dots$

Properties of the Gamma-function

① Γ - meromorphic in $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \dots$ (and no zeroes)

② $\Gamma(z+1) = z\,\Gamma(z)$, $\quad \Gamma(n) = (n-1)!$

$$\operatorname{Res}_{z=-n} \Gamma(z)\,dz = \frac{(-1)^n}{n!}, \qquad \Gamma(z+n) = \frac{\Gamma(z+n+1)}{z+n}$$

③ $$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\lim_{n\to\infty} z\left(\frac{1+z}{1}\right)\left(\frac{2+z}{2}\right)\cdots\left(\frac{n+z}{n}\right) n^{-z} \;\times$$

$$\lim_{n\to\infty} \left(\frac{1-z}{1}\right)\left(\frac{2-z}{2}\right)\cdots\left(\frac{n-z}{n}\right)(n+1-z)\, n^{z-1} =$$

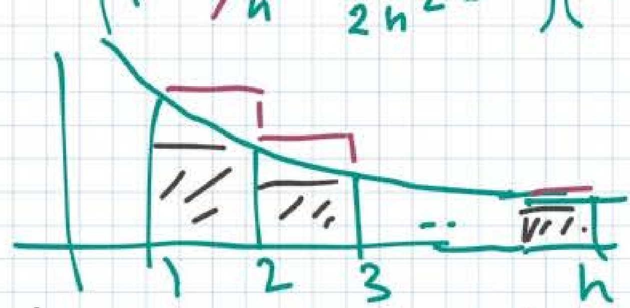
$$\underbrace{\lim_{n\to\infty} z(1-z^2)\left(1-\frac{z^2}{2^2}\right)\cdots\left(1-\frac{z^2}{n^2}\right)}_{=\sin \pi z/\pi} \; \underbrace{\lim_{n\to\infty} \frac{n+1-z}{n}}_{=1}$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \left[= \int_0^\infty e^{-t}\, t^{-1/2}\, dt \underset{t=u^2}{=} 2\int_0^\infty e^{-u^2}\, du\right]$$

The Weierstrass Infinite Product

$$g_n(z) = z \prod_{k=1}^{n} e^{-z/k}\left(1+\frac{z}{k}\right) \cdot e^{z\left[1+\frac{1}{2}+\dots+\frac{1}{n} - \log n\right]}$$

$$\left(1 - \cancel{\frac{z}{n}} + \frac{z^2}{2n^2} - \dots\right)\left(1 + \cancel{\frac{z}{n}}\right) = 1 - \frac{z^2}{2n^2} + \dots$$



decreasing, > 0

$\Rightarrow \lim_{n\to\infty} =: C$

$$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \le \log n \le 1 + \frac{1}{2} + \dots + \frac{1}{n-1}$$

$$\frac{1}{\Gamma(z)} = z\, e^{Cz} \prod_{k=1}^{\infty} e^{-z/k}\left(1+\frac{z}{k}\right)$$

Euler's constant, ≈ 0.5

$$\frac{\Gamma'}{\Gamma} = -\left[\frac{1}{z} + C + \sum_{n\ge 1}\left(\frac{1}{z+n} - \frac{1}{n}\right)\right]$$

Corollary: For $x > 0$, $\log \Gamma(x)$ is convex.

$$(\log \Gamma)'' = \sum_{n\ge 0} \frac{1}{(x+n)^2} > 0$$

Compactness in $\mathcal{C}(D) \supset \mathcal{H}(D)$

Def. K - compact $\Longleftrightarrow$ every open cover has a finite subcover.

$\Downarrow \quad \cancel{\Uparrow}$

Remark - Exercise in metric spaces.

Sequential compactness $\Longleftrightarrow$ every sequence has a convergent subsequence

① Compact $\Rightarrow$ Bounded — $d(x_n, x_0) > n \Rightarrow \{x_n\}$ has no convergent subsequence.

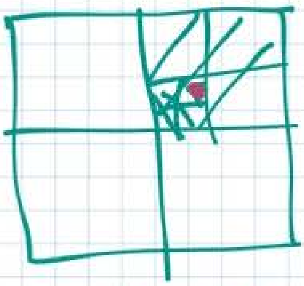
② Compact $\Rightarrow$ Closed — $x_n \to x_0 \notin K \Rightarrow \{x_n\}$ has no subsequence convergent in K.

③ $F \subset K \Rightarrow F$ - compact.
closed, compact

$\{f_n\}$ has a subseq. $\{f_{n_k}\}$ conv. in K, hence in F

④ Heine-Borel's Thm: A closed bounded subset in $\mathbb{R}^n$ is compact.

Proof:



Pick one of 2^n cubes which contains infinitely many terms of the sequence.

• - common point of selected cubes is a subsequential limit.

Counter-example: $\{\sin nx,\ n = 1, 2, \ldots\}$ does not have a uniformly conv. subsequence

Equicontinuity:

$\mathcal{C}(K) := \{f : K \to \mathbb{C}\}$

K – compact metric space; $\to$ – continuous

$\|f\| := \max_{x \in K} |f(x)|$

Def. $\{f\}$ is equicontinuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ for all functions f in the family.

Exercise: A non-equicontinuous family contains a sequence which has no convergent subsequence.

Thus: For compactness of $\{f\} \subset \mathcal{C}(K)$ it is necessary that $\{f\}$ is equicontinuous

Urzela-Ascoli's Theorem

A bounded equicontinuous sequence $f_n \in \mathcal{C}(K)$ of continuous functions on a compact metric space K contains a uniformly convergent subsequence.

① **Lemma** A pointwise bounded sequence $f_n : \mathbb{N} \to \mathbb{C}$ contains a pointwise-converging subsequence

$\{f_1\}$: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ ... subsequence conv. at $x=1$

$\{f_2\}$: $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ ... subseq. of $\{f_1\}$ conv. at $x=2$

$\{f_3\}$: $f_{3,1}$ $f_{3,2}$ $f_{3,3}$ ... Subseq. of $\{f_2\}$ conv. at $x=3$

$\vdots$ ... $\ddots$ — " —

$\Rightarrow \{f_{n,n}\}$ converges at all $x = 1, 2, \dots, n, \dots$

② A compact K contains a countable dense subset.

K does not have a sequence $\{x_k\}$ with all $d(x_k, x_\ell) \geq \varepsilon$. ("$\varepsilon$-net")

For $\varepsilon = \frac{1}{n}$, pick a maximal set $x_1^{(n)}, x_2^{(n)}, \dots, x_{N(n)}^{(n)}$ with this property. Then $\{x_i^{(n)}\}$ - dense.

③ Pick a subsequence $\{f_{n_k}\}$ convergent pointwise at $\{x_i^{(n)}\}$ (boundedness of $\{f_n\}$)

Then $\{f_{n_k}\}$ is uniformly Cauchy.

Equicontinuity: $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$d(x, x') < \delta \Rightarrow |f_{n_k}(x) - f_{n_k}(x')| < \varepsilon/3 \ \forall k$

Let $x_1^{(n)}, \dots, x_{N(n)}^{(n)}$ be a δ-net (i.e. $\frac{1}{n} < \delta$)

Then for k, ℓ large enough ($>$ some M)

$|f_{n_k}(x_i^{(n)}) - f_{n_\ell}(x_i^{(n)})| < \varepsilon/3$

Then for all $x \in K$, $|f_{n_k}(x) - f_{n_\ell}(x)| \leq$

$|f_{n_k}(x) - f_{n_k}(x_i)| + |f_{n_k}(x_i) - f_{n_\ell}(x_i)| + |f_{n_\ell}(x_i) - f_{n_\ell}(x)|$

④ $\{f_{n_k}\}$ uniformly Cauchy $\Rightarrow \lim_{k\to\infty} f_{n_k} \in \mathcal{C}(K)$

Completeness of $\mathcal{C}(K), \|\cdot\|$.

Compact subsets in $\mathcal{H}(D)$

Fundamental Theorem: A sequence $\{f_n\}$ in $\mathcal{H}(D)$ uniformly bounded on compact subsets of D contains a subsequence converging in $\mathcal{H}(D)$ uniformly on compact subsets.

Remark: $\forall$ compact $K \subset D$ $\exists M$ s.t. $\|f_n\|_K \le M$ $\forall n$.

Corollary: A subset in $\mathcal{H}(D)$ is compact if and only if it is closed and bounded (uniformly on compact subsets in D).

1 st proof of the Fundamental Theorem

Inevitably, a sequence $\{f_n\}$ in $\mathcal{H}(D)$ bounded on compacts in D must be equicontinuous on compacts in D.

It suffices to prove this for $K = \overline{B(z,r)} \subset D$ for each $z \in D$ and sufficiently small $r = r(z)$ [Any compact is covered by finitely many of those].

$$|f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} f'(z)\,dz \right| \le |z_2 - z_1| \max_{z \in [z_1, z_2]} |f'(z)|$$

if f is holom. in a disk containing z_1, z_2.

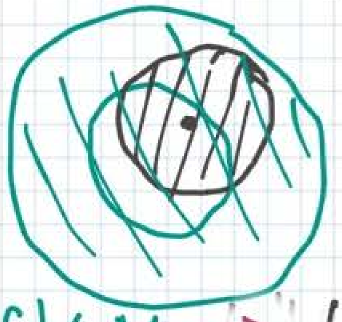
$\Rightarrow$ Suffices to prove that f_n' are bounded on compact subsets in D.

Lemma: $|f(z)| \le M$ for $|z| \le 2r$ [f is holomorphic]

$$\Rightarrow |f'(z)| \le \frac{M}{r} \text{ for } |z| \le r$$

Cauchy's inequality

$$\left[f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^2} \right]$$



$|f| \le M \Rightarrow |f'| \le \frac{M}{r}$

2nd Proof of the Fundamental Thm

Proposition $\{f_n\}$ holom. in $|z|<\rho$ and bounded on compact subsets converge uniformly on compact subsets if and only if for each $k=0,1,2,..,$ $\{\frac{d^k f}{dz^k}(0)\}$ converge (i.e. all Taylor coeff. at $z=0$ converge).

On $|z|\le r_0 (<\rho)$, $|f_n(z)|\le M$ for all n.

$$f_n(z)=\sum_{k\geq 0} a_{n,k} z^k, \quad |a_{n,k}|\le \frac{M}{r_0^k}$$

Cauchy inequalities.

$\Rightarrow$ for $|z|\le r<r_0$

$$|f_m(z)-f_n(z)|\le \sum_{k\le k_0} |a_{m,k}-a_{n,k}|r^k + 2M\sum_{k>k_0}\left(\frac{r}{r_0}\right)^k$$

(2) $\le \varepsilon/2$ for $m,n\geq N$ with N large enough

(1) $\le \varepsilon/2$ for k_0 large enough

$\Rightarrow$ $\{f_n\}$ is Cauchy in $|z|\le r$ $\Rightarrow$ converges uniformly on compact subsets to a holom. f. on $|z|<\rho$.

Corollary: A sequence $\{f_n\}$ holom. in $|z|<\rho$ and bounded on compact subsets contains a subsequence converging uniformly on compact subsets.

Indeed, sequence of functions $\varphi_n : \mathbb{N}\to\mathbb{C}$
$\varphi_n(k)=\frac{d^k f_n}{dz^k}(0)$ is pointwise bounded
$\Rightarrow$ contains a pointwise-convergent subsequence.
[Lemma proved by the diagonal argument.]

Application of the Fund. Theorem.

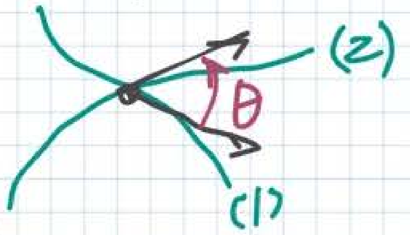
If $\{f_n\}$ in $\mathcal{H}(D)$ is bounded on compact subsets and converges pointwise, then it conv. in $\mathcal{H}(D)$.

Pointwise $\lim f_n = \lim f_{n_k}$ $\leftarrow$ conv. in $\mathcal{H}(D)$

So the limit pt. of $\{f_n\}$ in $\mathcal{H}(D)$ exists and is unique.

Conformal mappings

Angle between parametric curves



= angle between their (non-zero) velocity vectors

Conformal mapping:

$$\mathbb{C} \supset U \xrightarrow{f} V \subset \mathbb{C}$$

differentiable mappings preserving angles between curves.

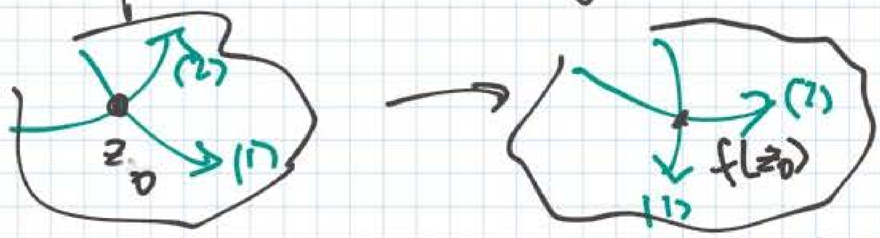
Examples: (1) Among linear maps:

$w = (a+bi)z$ $\qquad$ $w = (a+bi)\bar{z}$

rotation & expansion $\qquad$ & reflection

$a+bi \neq 0$ $\qquad$ (~~$a \neq 0 \neq b$~~)

(2) $f: U \to V$ - holom., $f' \neq 0$



linear approximation $\qquad \Delta w = f'(z_0)\Delta z$

rotation & expansion.

Inverse Function Theorem

$f'(z_0) \neq 0 \Leftrightarrow f$ is a local homeo/diffeomorphism (and ∞-diff.)

(3) $U \to V$ - anti holom.,

$z \mapsto f(\bar{z})$ holomorphic, $f' \neq 0$

Proposition: A conformal mapping on a connected domain U is either holom. or anti-holomorphic, with non-vanishing Jacobian.

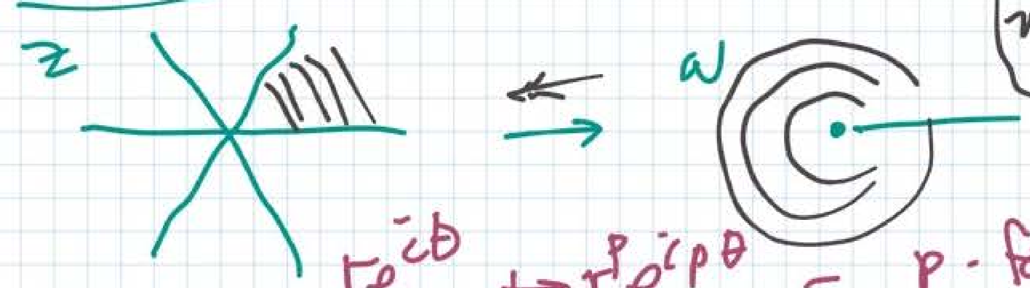
Linearization: either $\frac{\partial f}{\partial \bar{z}} = 0$ or $\frac{\partial f}{\partial z} = 0$, but not both. Jacobian $a^2+b^2 > 0$ $\qquad -a^2-b^2 < 0$

$\Rightarrow$ everywhere $\gtrless 0$ (connectedness)

Local study of $w = f(z)$ when $f'(z_0) = 0$

Example: $w = z^p$, $p > 1$

$\boxed{z = w^{1/p} \text{ multiple-valued}}$



$re^{i\theta} \mapsto r^p e^{ip\theta}$ – p-fold map

In general: $w = \underset{\neq 0}{c}\, z^p (1 + a_1 z + a_2 z^2 + \dots)$ — p-fold critical point

$\Rightarrow w = [c^{1/p} z (1 + a_1 z + \dots)^{1/p}]^p$

$c^{1/p}$: p choices; $(\dots)^{1/p}$: principal branch

$w = g(z)^p : U \to V \to$

$z \mapsto g(z)$ (locally invertible), $g \mapsto g^p$ (example)

$g'(0) = c^{1/p} \neq 0$

Theorem: $f(D)$ is open (unless one point)

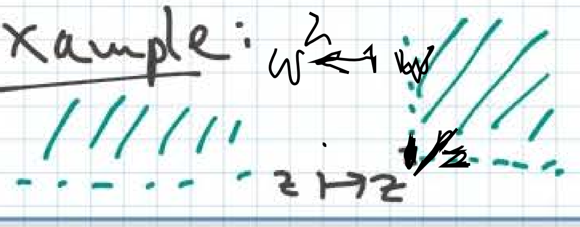
f holomorphic, D connected

Proof: $f(z_0)$ is contained in $f(D)$ together with a nbhd – if z_0 is a non-critical point (by the inverse function theorem), and if $p > 1$, because this is true for g and for the example ($g \mapsto g^p$).

Corollary: If $f: U \to \mathbb{C}$ – is injective holom. function, then it is a ~~homeomorphism~~ isomorphism $U \cong f(U)$. Indeed, since f is open f^{-1} is continuous. injective $\Rightarrow f' \neq 0$

Def. Isomorphism $U \to V$
– holom. bijection whose inverse is holom.
$\Leftrightarrow$ holom & injective

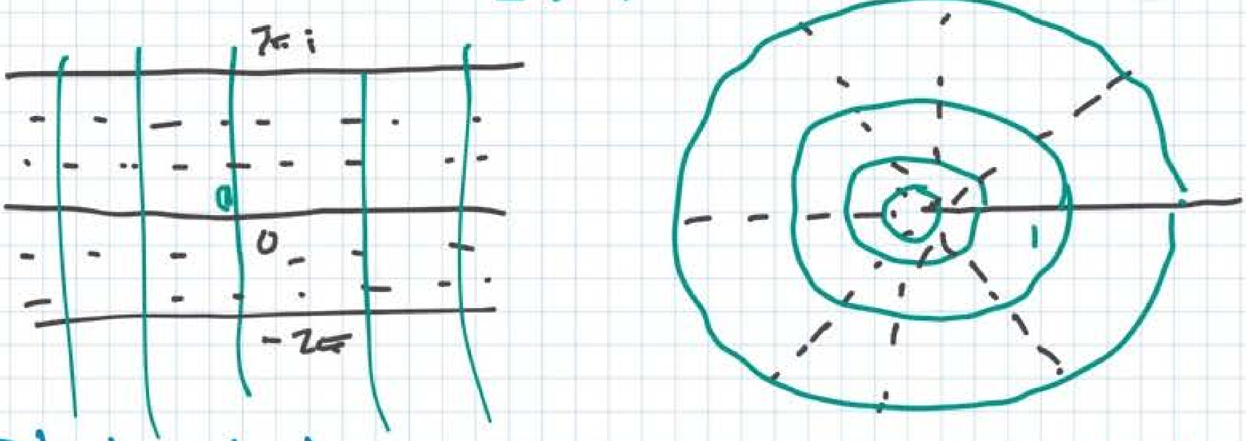
Example:



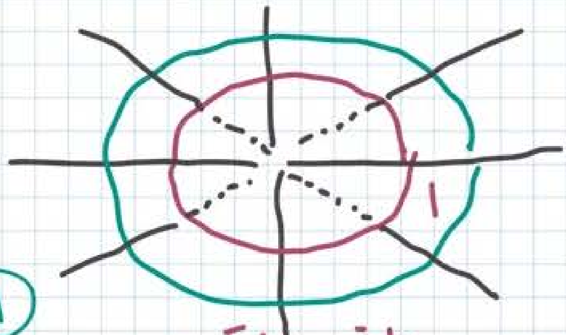
Remark: All concepts make sense for $\mathbb{C}P^1 \supset U \to V \subset \mathbb{C}P^1$

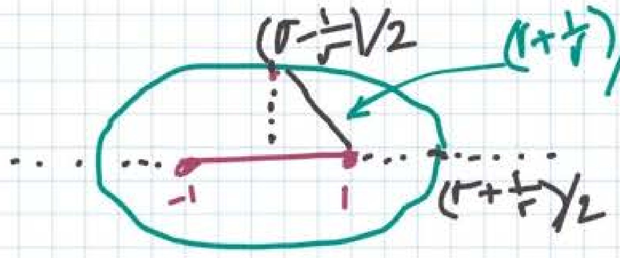
Examples of conformal mappings

$$z \longmapsto w = \exp z$$



Zhukovsky's function: $z \longmapsto w = \frac{1}{2}\left(z + \frac{1}{z}\right)$





① $t \mapsto \frac{1}{2}(e^{it} + e^{-it}) = \cos t$

② $f(z) = f\left(\frac{1}{z}\right)$

③ $|z| = r > 1$

$$\frac{re^{it} + \frac{1}{r}e^{-it}}{2} = \overbrace{\underbrace{\left(\frac{r+\frac{1}{r}}{2}\right)}_{a}\cos t}^{x} + i\,\overbrace{\underbrace{\left(\frac{r-\frac{1}{r}}{2}\right)}_{b}\sin t}^{y}$$

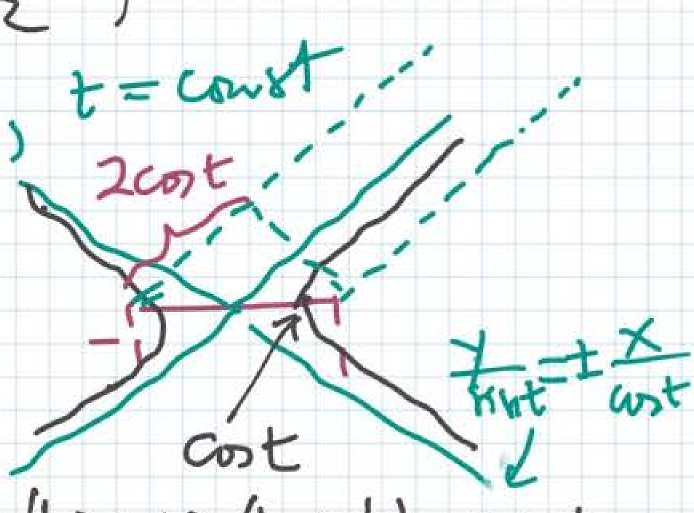
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse with foci ± 1.

$$\left(\frac{r-\frac{1}{r}}{2}\right)^2 + 1^2 = \left(\frac{r+\frac{1}{r}}{2}\right)^2$$

④ $|z| > 1$, $z = re^{it}$, $t = \text{const}$

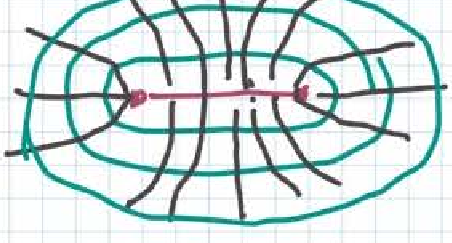
$$\frac{x^2}{(\cos t)^2} - \frac{y^2}{(\sin t)^2} = 1$$

hyperbola with foci ± 1



$$(1+\cos t) - (1-\cos t) = \underline{2\cos t}$$

⑤ Confocal ellipses and hyperbolas are pairwise orthogonal



$$z \longmapsto iz \xmapsto{\exp} e^{iz} \longmapsto \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Zhukovsky's function

The Möbius Group

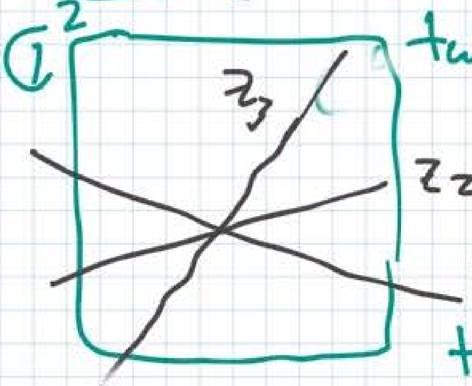
$$PGL_2(\mathbb{C}) = \left\{ w = \frac{az+b}{cz+d} \;\middle|\; ad \neq bc \right\}$$

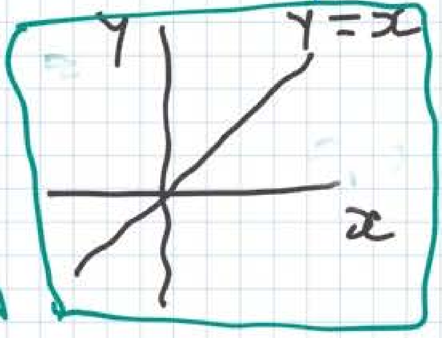
- automorphisms of $\mathbb{CP}^1 = \left\{ \begin{array}{c}\text{1-dim} \\ \text{subspaces} \\ \text{in } \mathbb{C}^2\end{array} \right\}$

$$= GL_2(\mathbb{C})/\text{center} = \frac{\left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \det \neq 0 \right\}}{\left\{ \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right],\ \lambda \neq 0 \right\}}$$

$$= SL_2(\mathbb{C})/(\pm I) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \det = 1 \right\} / (\pm I)$$

Theorem: It acts triply-transitively on $\mathbb{CP}^1$





$\mathbb{CP}^1 \ni z_1, z_2, z_3$ distinct $\rightsquigarrow 0, 1, \infty \in \mathbb{CP}^1$

Corollary 1: No other automorphisms of $\mathbb{CP}^1$

$$\mathbb{CP}^1 \xrightarrow{f} \mathbb{CP}^1 \xrightarrow{\text{möbius}} \mathbb{CP}^1$$
$$\infty \mapsto z \mapsto \infty$$

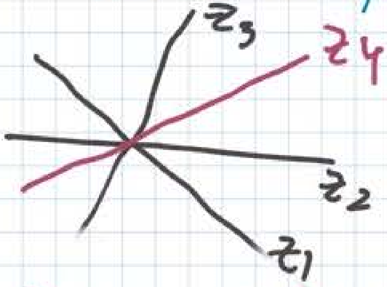
$g(z) = a_0 + a_1 z + \cancel{\dots + a_n z^n}$ one-to-one

pole at ∞

Corollary 2 $(z_1, z_2, z_3, z_4) \rightsquigarrow (0, 1, \infty, \lambda)$, $\lambda \neq 0, 1, \infty$

distinct

$\lambda(z_1, z_2, z_3, z_4)$ is Möbius invariant!



$$X := y - z_3 x = \alpha(y - z_1 x) =: Y$$

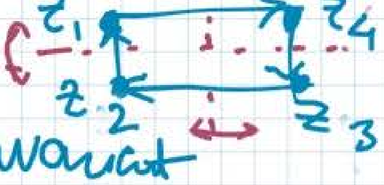
on $y = z_2 x$

$$\Longrightarrow \alpha = \frac{z_2 - z_3}{z_2 - z_1}$$

$$\lambda = \frac{\alpha(y - z_1 x)}{(y - z_3 x)}\Bigg|_{y = z_4 x} = \boxed{\frac{(z_2 - z_3)(z_4 - z_1)}{(z_2 - z_1)(z_4 - z_3)}}$$

"cross-ratio"

Corollary 3



λ is K_4-invariant

$$S_4 / K_4 \simeq S_3$$

$S_{(0,1,\infty)}$ $\lambda \mapsto 1/\lambda$, $\lambda \mapsto 1 - \lambda$

$$\lambda \mapsto 1/\lambda \mapsto \frac{\lambda - 1}{\lambda} \mapsto \frac{\lambda}{\lambda - 1} \mapsto \frac{1}{1 - \lambda} \mapsto 1 - \lambda \mapsto \lambda$$

Möbius invariance of "clircles" (34.1)

Thm Fractional-linear transformations map "clircles" (:= lines or circles) into "clircles".

① $\frac{az+b}{0\cdot z+d} = Az + B$ ← translation

$d \neq 0$ ↑ rotation/expansion

$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{(ad-bc)}{c(cz+d)} = A + \frac{B}{z+C}$ ← inversion

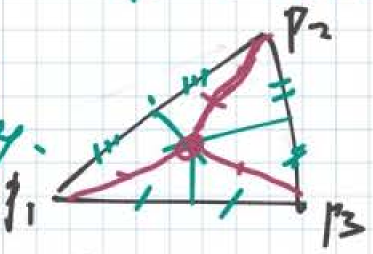
$c \neq 0$; A: translation; C: translation

② "clircles" $\alpha|z|^2+\beta \operatorname{Re} z+\gamma \operatorname{Im} z+\delta=0$

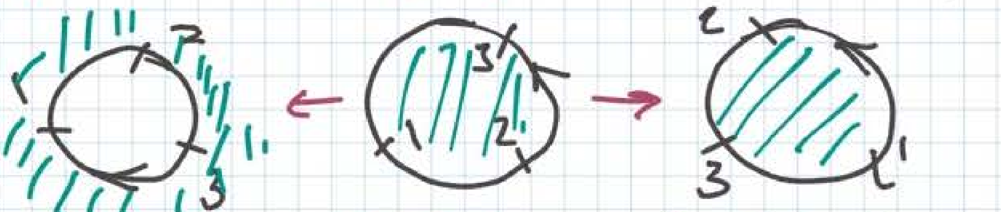
$z\mapsto \frac{1}{z}$: $\frac{1}{|z|^2}\left(\alpha+\beta \operatorname{Re} z-\gamma \operatorname{Im} z+\delta|z|^2\right)=0$

Corollary. Every (C, p_1, p_2, p_3) ← clircle; three distinct points on C — can be transformed into any (C', p_1', p_2', p_3') by a unique fractional-linear transformation.

- In $\mathbb{C}P^1$, through any distinct (p_1, p_2, p_3) there is a unique clircle.
- $PGL_2(\mathbb{C})$ acts triply-transitively (mapping clircles to clircles).

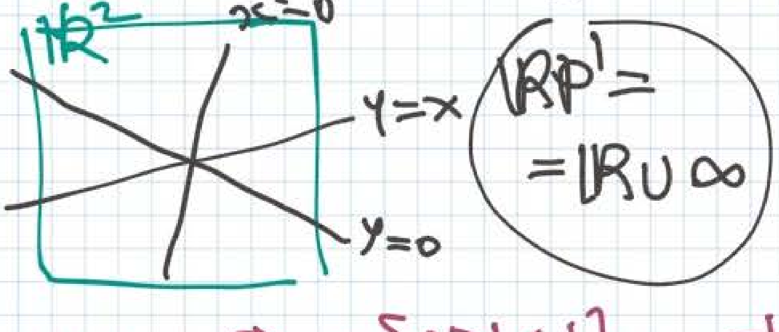


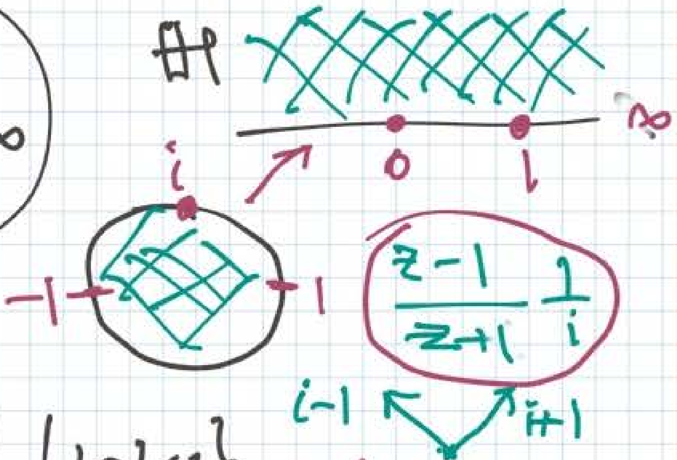
Remark:



Example: $\operatorname{Aut}(\mathbb{H})$ ← upper half-plane $\operatorname{Im} z>0$

$= PGL_2^{(+)}(\mathbb{R}) = \left\{ \frac{az+b}{cz+d} \;\middle|\; \begin{array}{l} ad-bc>0 \\ a,b,c,d\in\mathbb{R}\end{array} \right\} = \frac{SL_2(\mathbb{R})}{\pm I}$





$\mathbb{H} \simeq \mathbb{D} = \{|z|<1\}$

$\operatorname{Aut}(\mathbb{D}) = \left\{ e^{i\theta}\frac{(z-a)}{(1-z\bar{a})} \;\middle|\; |a|<1 \right\} \subset \operatorname{Aut}(\mathbb{C}P^1)$

Schwarz lemma

"Fundam. Theorem on Conformal Representation" = the classical Riemann Mapping Theorem

<u>Theorem</u> A connected simply-connected open subset in $\mathbb{CP}^1$ is isomorphic to exactly one of $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{D} := \{|z|<1\}$

(1) All three are connected, simply-connected

(2) They are pairwise non-isomorphic

<u>Proof 1</u>. $\mathbb{CP}^1$ is compact, $\mathbb{C}, \mathbb{D}$ are not.

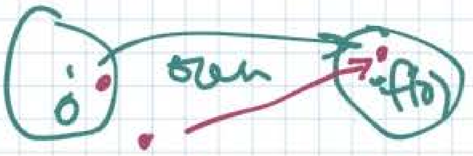
$\mathbb{C} \to \mathbb{D}$ is constant (Liouville)
$\Rightarrow$ not bijective.

<u>Proof 2</u>. $\mathrm{Aut}(\mathbb{CP}^1) = PGL_2(\mathbb{C})$

$\mathrm{Aut}(\mathbb{D}) \simeq \mathrm{Aut}(\mathbb{H}) = PGL_2(\mathbb{R})$ (*)

<u>Lemma</u>: $\mathrm{Aut}(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{CP}^1)$

<u>Proof</u>: An entire function $\mathbb{C} \xrightarrow{f} \mathbb{C}$ with an essential singularity at ∞ cannot be one-to-one:



By Weierstrass' Theorem $f(|z|>\varepsilon)$ is dense.

$\Rightarrow \mathrm{Aut}(\mathbb{C}) = \{w = az+b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$

$\dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{CP}^1) = 6$ triply-transitive

$\dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{C}) = 4$ doubly-transitive

$\dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{D}) = 3$ transitive + ?

(3) $\mathbb{C} \neq D$ - simply connected $\simeq \mathbb{D}$ (connected)

<u>Corollary</u> Every simply-connected subset of $\mathbb{R}^2$ is <u>homeomorphic</u> to $\mathbb{R}^2$.

Proof of the classical Riemann Thm

Theorem: A simply connected open $D \subsetneq \mathbb{C}$ is isomorphic to $\mathbb{U} = \{|z|<1\}$.

Step 1: WLOG, D is bounded.

$$a \notin D \underset{\exp}{\overset{}{\rightleftarrows}} D'$$

$g: z \mapsto \log(z-a)$ – single-valued branch exists since D is simply-connected

injective because exp is a function

disjoint since exp is $2\pi i$-periodic

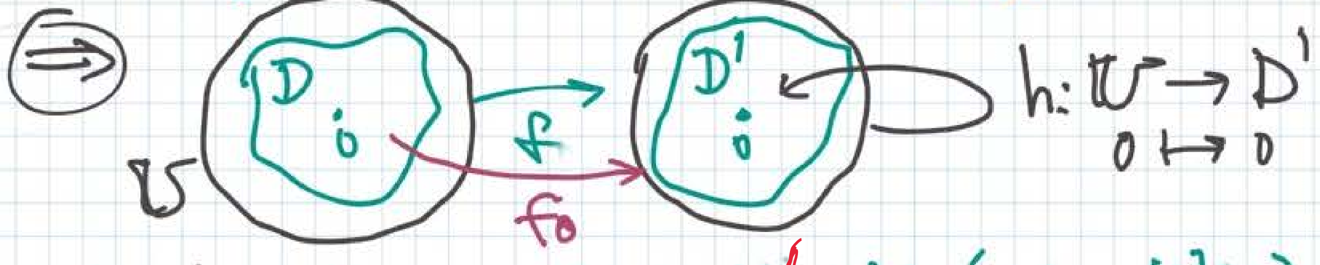
$$\Rightarrow \left| \frac{1}{g(z)-g(z_0)-2\pi i} \right| < \frac{1}{\varepsilon}$$

Corollary: We may assume that $0 \in D \subset \mathbb{U}$.

Step 2. $\mathcal{A} := \{ f : D \to \mathbb{U} \text{ – injective}, f(0)=0 \}$

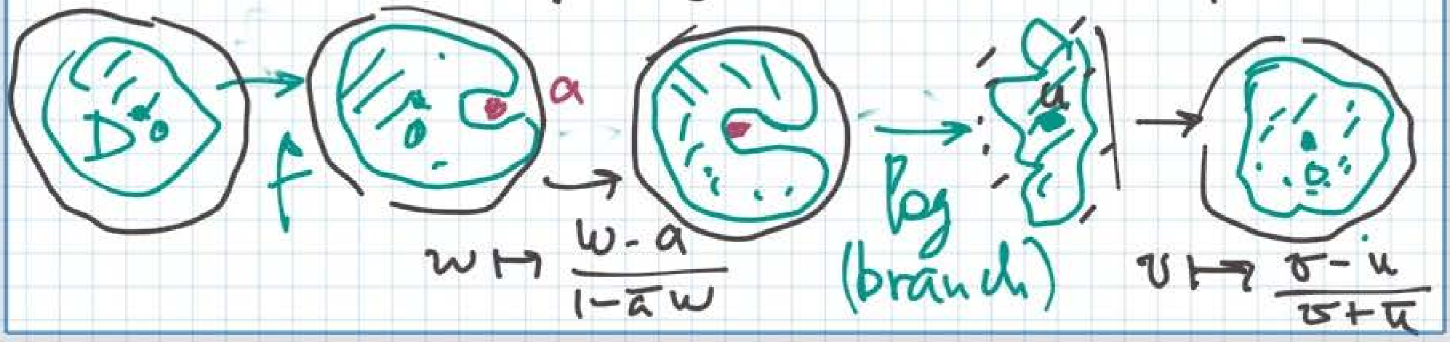
Then $f_0(D) = \mathbb{U} \iff |f_0'(0)| = \max_{f \in \mathcal{A}} |f'(0)|$

($\Rightarrow$)



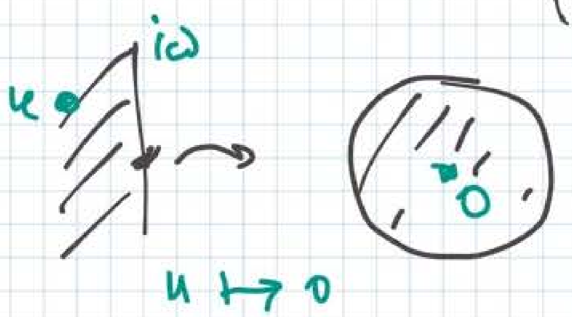
$f = h \circ f_0$ $\quad |f'(0)| = \underbrace{|h'(0)|}_{\le 1 \text{ – Cauchy}} |f_0'(0)| \le |f_0'(0)|$

($\Leftarrow$) Contra-positive: If $f(D) \subsetneq \mathbb{U}$, construct $\tilde{f} \in \mathcal{A}$ with $|\tilde{f}'(0)| > |f'(0)|$.



In formulas $\boxed{\text{Remember: } f(0)=0}$

$$z \mapsto F(z) := \log \frac{f(z)-a}{1-\bar a f(z)} \mapsto \tilde f(z) = \frac{F(z)-F(0)}{F(z)+\overline{F(0)}}$$



$$v \mapsto \frac{v-u}{v+\bar u}$$

$$v = i\omega : \left|\frac{i\omega - u}{i\omega+\bar u}\right| = 1$$

$$\tilde f'(0) = \frac{F'(0)}{F(0)+\overline{F(0)}} \qquad F'(0) = \frac{1-a\bar a}{-a} f'(0)$$

Lemma: $\left(\frac{az+b}{cz+d}\right)' = \left[\frac{a(\cancel{cz}+d)-(\cancel{az}+b)c}{(cz+d)^2}\right] = \frac{ad-bc}{(cz+d)^2}$

$$\frac{|\tilde f'(0)|}{|f'(0)|} = \frac{1-a\bar a}{2|a| \log \frac{1}{|a|}} > 1 \quad \left(\begin{matrix}\text{remember:}\\ |a|<1\end{matrix}\right)$$

$$\frac{1}{|a|} =: e^t, \; 0<t< \frac{e^t - e^{-t}}{2} = \sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$$

Step 3: Compactness

$$\mathcal{A}_1 = \{ \mathbb{D} \supset D, 0 \xrightarrow{f} \mathbb{C}, 0 \text{ injective}, |f'(0)| \geq 1 \}$$

non-empty: $f(z) \equiv z$

bounded: $|f(z)| < 1$ for all $z \in D$, $f \in \mathcal{A}$

$\left|\frac{d}{dz}\right|_{z=0}$ continuous $[f_n \to f \Rightarrow f_n' \to f' \Rightarrow |f_n'(0)| \to |f'(0)|]$

$\longrightarrow$ bounded $\left[\{|z|<\varepsilon\} \subset D \Rightarrow |f'(0)| \leq \frac{1}{\varepsilon}\right]$

Alternatively: $\mathcal{A}_1$ is closed in $\mathcal{H}(D)$

$f_n \to f \Rightarrow f_n(0) \to f(0) \Rightarrow f(0) = 0$

$f_n'(0) \to f'(0) \Rightarrow |f'(0)| \geq 1$. ($\Rightarrow f \neq \text{Const}$)

$|f_n(z)| < 1 \Rightarrow |f(z)| \leq 1 \Rightarrow < 1$ (maximum modulus principle)

f_n - injective $\Rightarrow f$ - injective (since non-constant)

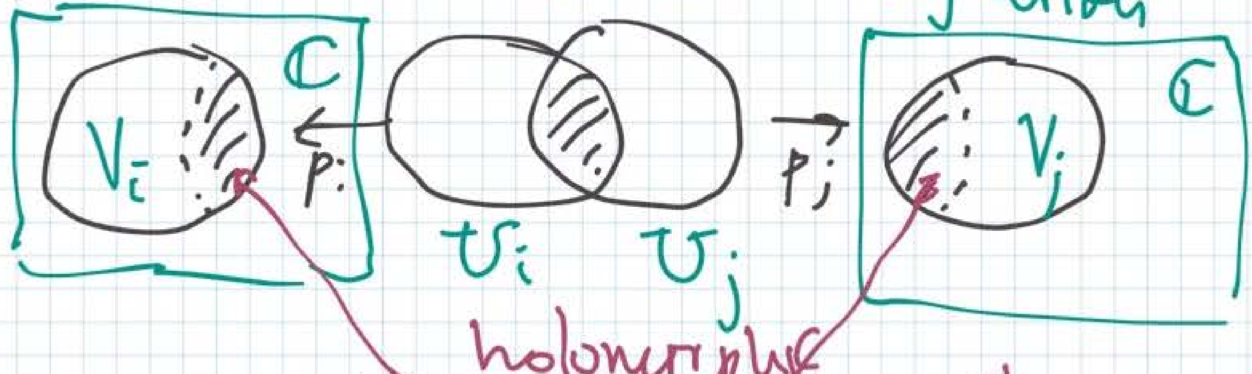
Thus, $\mathcal{A}_1$ is compact $\Rightarrow |f'(0)|$ achieves max.

Alternatively: $|f_n'(0)| \to \sup_{f \in \mathcal{A}_1} |f'(0)| \Rightarrow f_{n_k} \to f$.

One-dimensional Complex Manifolds

Set $M = \bigcup_i U_i$ $\qquad U_i \xrightarrow{p_i} V_i \in \mathbb{C}$

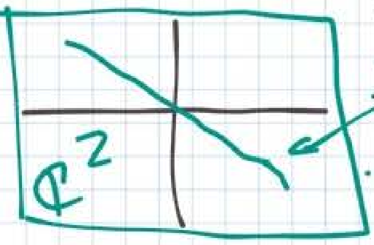
($\leq$ countably many) subsets $\qquad$ bijection $\qquad$ open sets



$p_j \circ p_i^{-1} \longleftarrow \longrightarrow p_j \circ p_i^{-1}$

Examples:

$\mathbb{C}P^1 \qquad \underset{z}{V_1 = \mathbb{C}} \qquad \underset{w}{\mathbb{C} = V_2}$



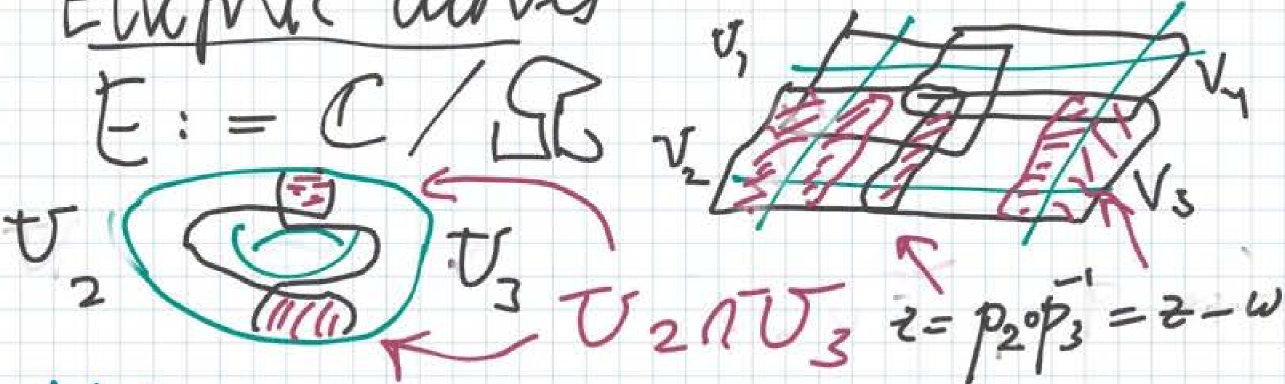
$U_1 \qquad U_2$

$\longrightarrow \; p_2 \circ p_1^{-1}(z) = \frac{1}{z}$

$\frac{1}{w} = p_1 \circ p_2^{-1}(w) \longleftarrow$

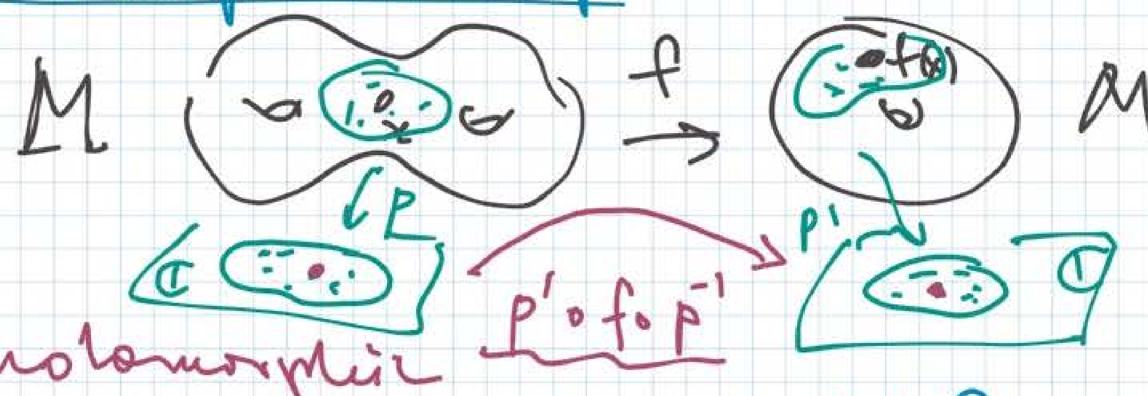
Elliptic curves

$E := \mathbb{C}/\Omega$

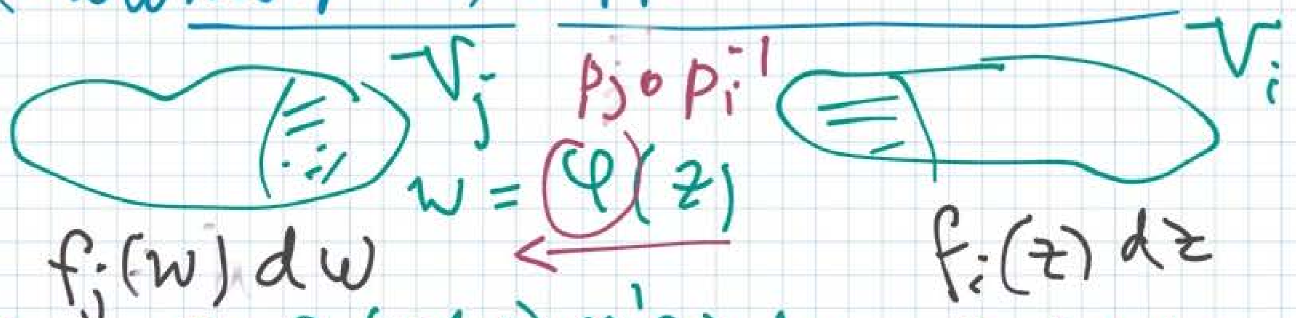


$z = p_2 \circ p_3^{-1} = z - \omega_1$

Holomorphic maps



(Holomorphic) differential 1-forms

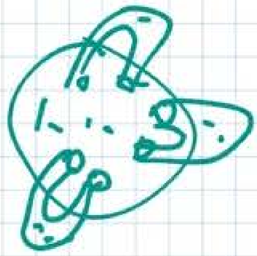


On $\quad f_j(\varphi(z))\,\varphi'(z)\,dz = f_i(z)\,dz$

Compact Complex 1-dim manifolds

Compact smooth orientable real surfaces:

"spheres with $g \geq 0$ handles



Liouville's Thm:

$M \xrightarrow{f} \mathbb{C} \Rightarrow f = \text{const}$

compact connected; holomorphic

Proof: max $|f|$ principle.

Residue Thm

$$\oint_{\partial M} \omega = 2\pi i \sum \operatorname{Res} \omega = 0 \quad \text{if } \partial M = \emptyset$$

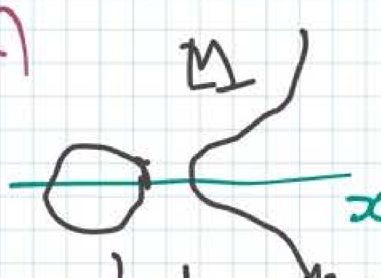
meromorphic 1-form

Riemann's Mapping Theorem

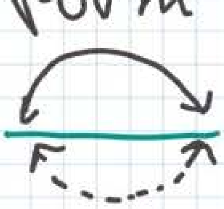
A connected simply-connected complex one-dim. mfd is biholomorphic to $\mathbb{CP}^1$, $\mathbb{C}$, or $\mathbb{U}$.

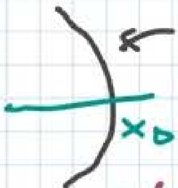
Thm: A non-singular cubical curve in $\mathbb{CP}^2$ is $\mathbb{C}/\Lambda$

$$y^2 = 4x^3 - 20a_2 x - 28a_4 =: p(x)$$



Lemma: $\omega := \frac{dx}{y}$ is a non-vanishing holomorphic 1-form on M.

①  $\omega = \frac{dx}{\sqrt{P(x)}}$

② 

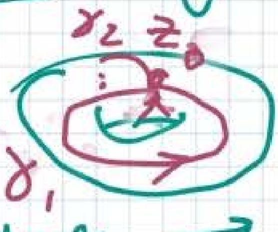
$x - x_0 = y^2 + \dots$ (IFT; loc. coord.)

$$\frac{dx}{y} = \frac{\cancel{y}(2 + \dots)dy}{\cancel{y}}$$

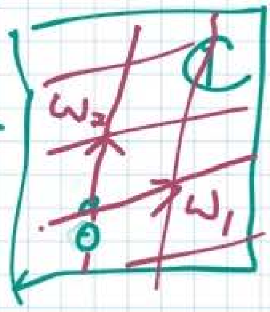
③ At ∞ - homework

Proof of Theorem

$$\oint_{\gamma_i} \frac{dx}{y} = \omega_i$$



multivalued $\int_{z_0}^{z} \frac{dx}{y} \longrightarrow$

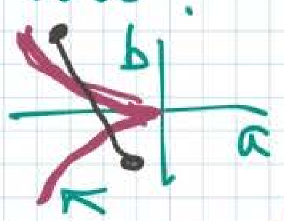


Some easy-to-miss details on cubics

① Why are non-singular cubics tori?

$$y^2 = P_3(x) = x^3 + ax + b$$

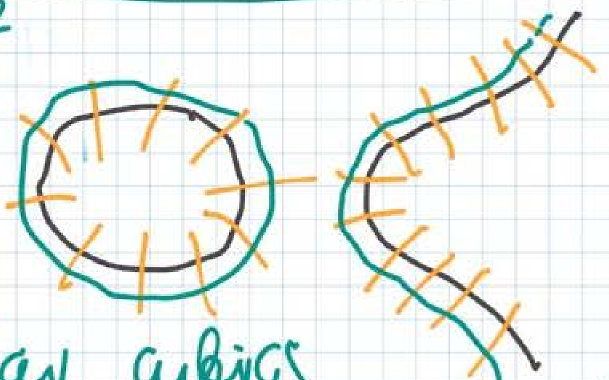
The space of non-sing. cubics is connected!



$$y^2 = x^3 - x \simeq \mathbb{C}/\mathbb{Z}^2$$

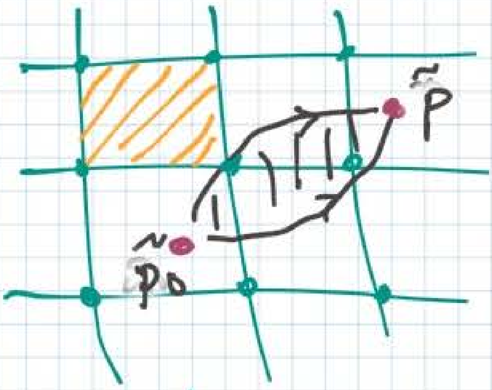
square lattice,

$\mathbb{CP}^2$

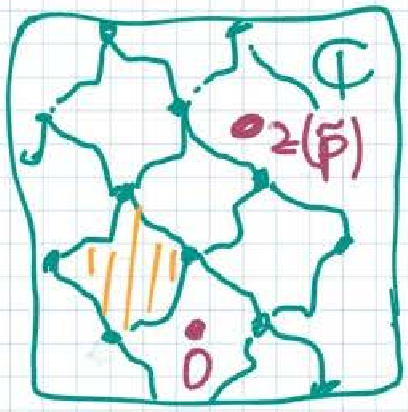


Nearby non-singular cubics are homeomorphic (why not iso?)

② $\mathbb{R}^2$



$$\xrightarrow{\int_{\tilde p_0}^{\tilde p} \omega}$$

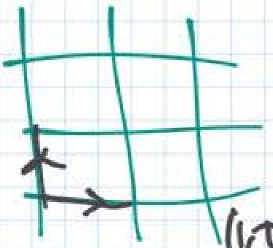


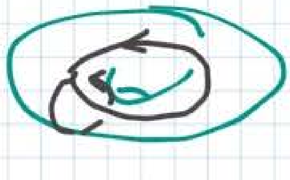
$\mathbb{R}^2/\mathbb{Z}^2$

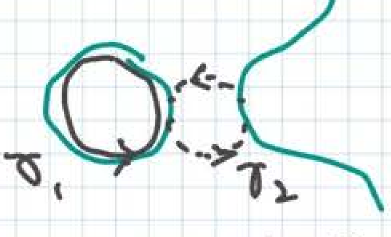
$\longrightarrow \mathbb{C}/\Omega$

Where do the complex structure and the holomorphic 1-form ω on $\mathbb{R}^2/\mathbb{Z}^2$ come from?

③ What exactly is the "period lattice" Ω?



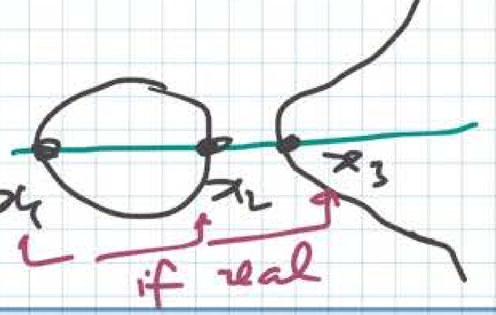




$$\Omega = \left\{ m \int_{(0,0)}^{(1,0)} \omega + n \int_{(0,0)}^{(0,1)} \omega \right\} = \left\{ m \oint_{\gamma_1} \frac{dx}{y} + n \oint_{\gamma_2} \frac{dx}{y} \right\}$$

$$\omega_1 = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{(x-x_1)(x-x_2)(x-x_3)}}$$

$$\omega_2 = 2 \int_{x_2}^{x_3} \frac{dx}{\sqrt{\quad '' \quad}}$$

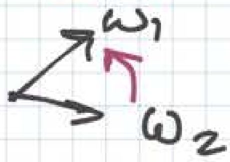


Classification of elliptic curves $\mathbb{C}/\Omega$

$\mathbb{C}/\Omega \simeq \mathbb{C}/\Omega'$ iff $\Omega' = c\,\Omega$, $c \neq 0$

Proof: $\Leftarrow$ obvious, $\Rightarrow$ ω is unique up to $c\times$

$\Omega = \{m\omega_1 + n\omega_2 \mid (m,n)\in\mathbb{Z}^2\}$ - lattice with a basis



is equivalent to

$\{m\tau + n\cdot 1 \mid (m,n)\in \mathbb{Z}^2\}$ $\quad \tau := \frac{\omega_1}{\omega_2} \in \mathbb{H}$ $\quad \operatorname{Im}\tau > 0$

Change of basis: $\omega_1' = a\omega_1 + b\omega_2$

$\omega_2' = c\omega_1 + d\omega_2$

$a,b,c,d \in \mathbb{Z}$

$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = \cancel{\pm} 1$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

modular group

$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I \subset \operatorname{Aut}(\mathbb{H}) = \frac{SL_2(\mathbb{R})}{\pm I}$

Columns form a right-oriented basis in $\mathbb{Z}^2$

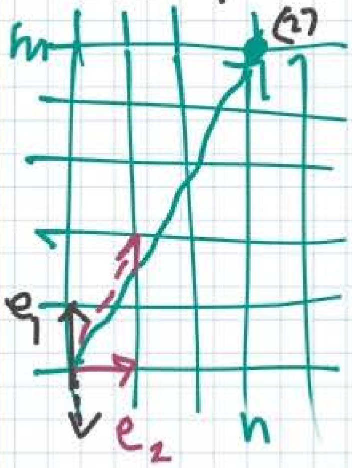
Proposition: Such a basis (e_1, e_2) can be made standard, $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$, by composition of transformations: $T: (e_1, e_2) \mapsto (e_1, e_2 + e_1)$

$S: (e_1, e_2) \mapsto (e_2, -e_1)$, or their inverses.

In other words, $(P)SL_2(\mathbb{Z})$ is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}$.

$\tau \mapsto \tau + 1$ $\qquad \tau \mapsto -1/\tau$

Proof: A Euclidean algorithm

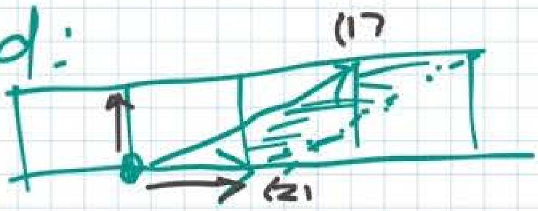


$\frac{m}{n} = q - \frac{r}{n}$ $\quad 0 \le r < n$ $\quad \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}^q$

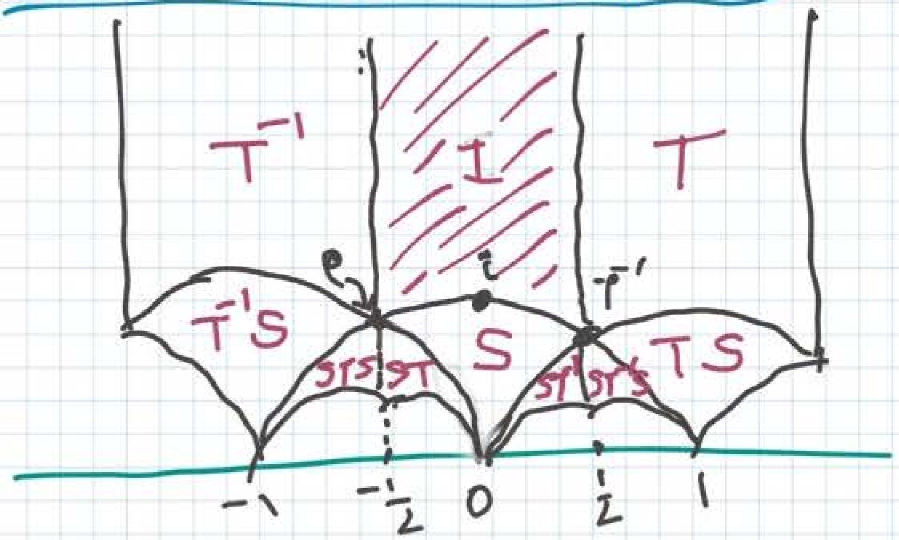
$\frac{n}{r} = q' - \frac{r'}{r}$ $\quad 0 \le r' < r$ $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$

. . . .

At the end:



The modular figure



$T: \tau \mapsto \tau + 1$

$S: \tau \mapsto -1/\tau$

generate $PSL_2(\mathbb{Z})$

$= \left\{ \frac{a\tau+b}{c\tau+d} \,\middle|\, ad-bc=1 \right\}$

$a,b,c,d \in \mathbb{Z}$

Remark: $\mathbb{H}/(T) \simeq \mathbb{D}\setminus 0 = \{ q = e^{2\pi i\tau}, \ 0<|q|<1 \}$

Theorem: is a fundamental domain

① $\operatorname{Im} \frac{a\tau+b}{c\tau+d} = \frac{\operatorname{Im}(ad\tau + bc\bar{\tau})}{|c\tau+d|^2} = \frac{\operatorname{Im}\tau}{|c\tau+d|^2}$

$(c,d) \mapsto |c\tau+d|^2$ – positive-definite quadratic form

$\Rightarrow \{(c,d) : |c\tau+d| \leq |\tau|^2\}$ is finite

$\forall \tau \ \exists \tau' = \frac{a\tau+b}{c\tau+d}$ with $\underline{\max}$ $\operatorname{Im}\tau'$, $-\frac{1}{2} \leq \operatorname{Re}\tau' \leq \frac{1}{2}$

and $|\tau'| \geq 1$ (otherwise $\operatorname{Im}(-1/\tau') > \operatorname{Im}\tau'$)

② Suppose $\tau' = \frac{a\tau+b}{c\tau+d}$, $\tau \in$

$\operatorname{Im}\tau' \geq \operatorname{Im}\tau \Rightarrow |c\tau+d| \leq 1$

$\Rightarrow |c| \leq 1$ (distance from to integers!)

(a) $c = 0$: $\tau' = \tau \pm b \Rightarrow \operatorname{Re}\tau = -\frac{1}{2} \overset{}{\underset{T^{\pm 1}}{\Leftrightarrow}} \operatorname{Re}\tau = \frac{1}{2}$

(b) $c = \pm 1$, $|c\tau+d| = 1$

$\Rightarrow$ either $|\tau| = 1$ $(d=0)$ $(\tau' = S\tau)$ or $\tau = \begin{cases} \rho \\ -\rho^{-1} \end{cases}$

c	d
1	1
-1	-1
1	-1
-1	1

③ $\frac{a\tau+b}{c\tau+d} = \tau \in$ $\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm I$

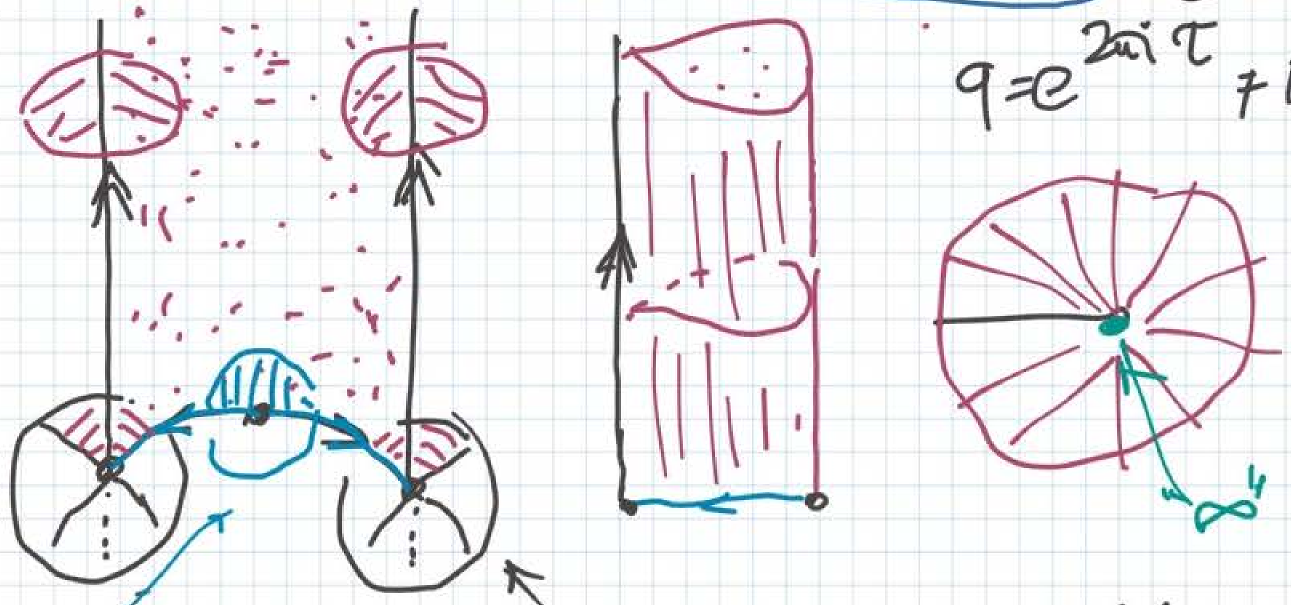
unless $\tau = i$ $(Si = i)$

or $\tau = \begin{cases} \rho \\ -\rho^{-1} \end{cases}$ $\begin{matrix} ST\rho = \rho \\ TS(-\rho^{-1}) = -\rho^{-1} \end{matrix}$

$(ST)^3 = I$

Complex manifold $\mathbb{H}/PSL_2(\mathbb{Z}) \simeq \mathbb{C}$] 38.2

$q = e^{2\pi i \tau} \neq 0$



$\mathbb{C} \to \mathbb{C}/\pm 1$

$z \mapsto w = z^2$

$\mathbb{C} \to \mathbb{C}/\{e^{\pm 2\pi i/3}, 1\}$

$z \mapsto w = z^3$

<u>Thm</u> $\underbrace{\mathbb{H}/PSL_2(\mathbb{Z})}_{\simeq \mathbb{C}} \sqcup \underbrace{\{q=0\}}_{\text{point at } \infty} \simeq \underbrace{\mathbb{CP}^1}_{\text{sphere}}$

<u>On the other hand:</u>

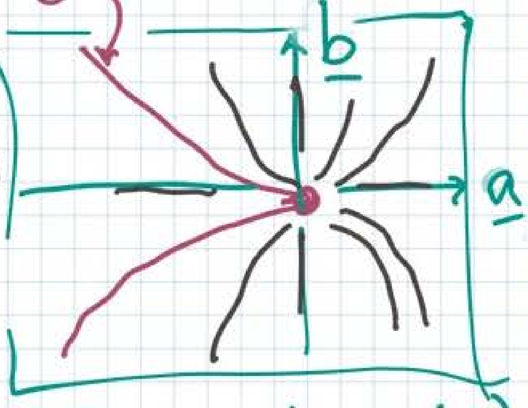
$$\mathbb{C}/\Omega \xrightarrow{\wp,\ \wp'} y^2 = 4x^3 - 20\, a_2 x - 28\, a_4$$

$a_2 = 3\sum_{\omega\neq 0} \frac{1}{\omega^4}$, $a_4 = 5\sum_{\omega \neq 0}\frac{1}{\omega^6}$

$/\lambda^4$ $/\lambda^6$

$\Omega \to \lambda\Omega$

$\Delta = 0$



$$J := \frac{a^3}{\underbrace{4a^3 + 27b^2}_{\Delta}} = \underline{\text{const}} \in \mathbb{CP}^1$$

$\neq 0$

discriminant

$(a,b) \mapsto \left(\frac{a}{\lambda^4}, \frac{b}{\lambda^6}\right)$

$$J: \mathbb{C}^2\setminus 0 \to \mathbb{C}^2\setminus 0/(\lambda) = \mathbb{CP}^1 = \underbrace{\frac{\mathbb{H}}{PSL_2(\mathbb{Z})}}_{\mathbb{C}\simeq} \sqcup \text{"}\infty\text{"}$$

"Orbifold" structure

Typical τ: $\mathbb{C}/\Omega \to \mathbb{C}/\Omega$, $z \mapsto -z$ $\quad \lambda = -1$

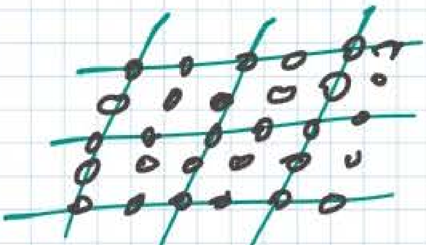
$\tau = i$: $z \mapsto iz$ (order 4) $\quad b = 0$, $\lambda = i$

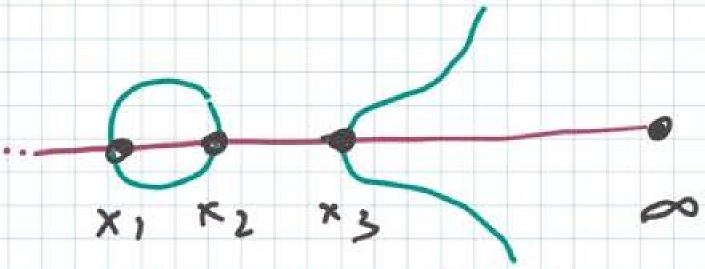
$\tau = \rho, -\rho^{-1}$: $z \mapsto e^{\pi i/3} z$ (order 6) $\quad a = 0$, $\lambda = e^{\pi i/3}$

Congruence-Subgroup $\Gamma(2) \subset PSL_2(\mathbb{Z})$

$$\mathbb{C}/\Omega \xrightarrow{\simeq} \{(x,y) \mid y^2 = x^3 + ax + b\}$$

$$\tfrac{1}{2}\Omega/\Omega \longmapsto \{(x,y) \mid y^2 = x^3 + ax + b,\ y = 0\}$$





"Weierstrass' points" of the elliptic curve

Problem: Classify elliptic curves up to isomorphisms respecting (the names of) Weierstrass points.

Rephrasing: In the lattice $\Omega = \{m\tau + n\cdot 1\}$ allow only chase of bases identical on $\tfrac{1}{2}\Omega/\Omega$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod 2$

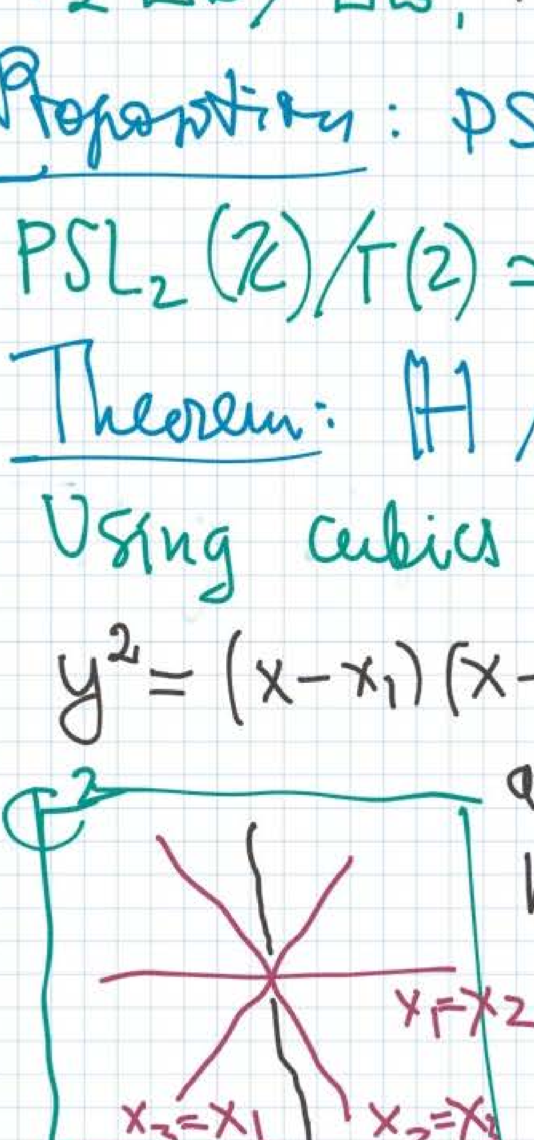
Proposition: $PSL_2(\mathbb{Z})/\Gamma(2) \simeq S_3$

$$PSL_2(\mathbb{Z})/\Gamma(2) \simeq GL_2(\mathbb{Z}_2) = \mathrm{Aut}(\mathbb{Z}_2^2)$$

Theorem: $\mathbb{H}/\Gamma(2) \simeq \mathbb{CP}^1 \setminus \{0, 1, \infty\}$

Using cubics in $\mathbb{CP}^2$:

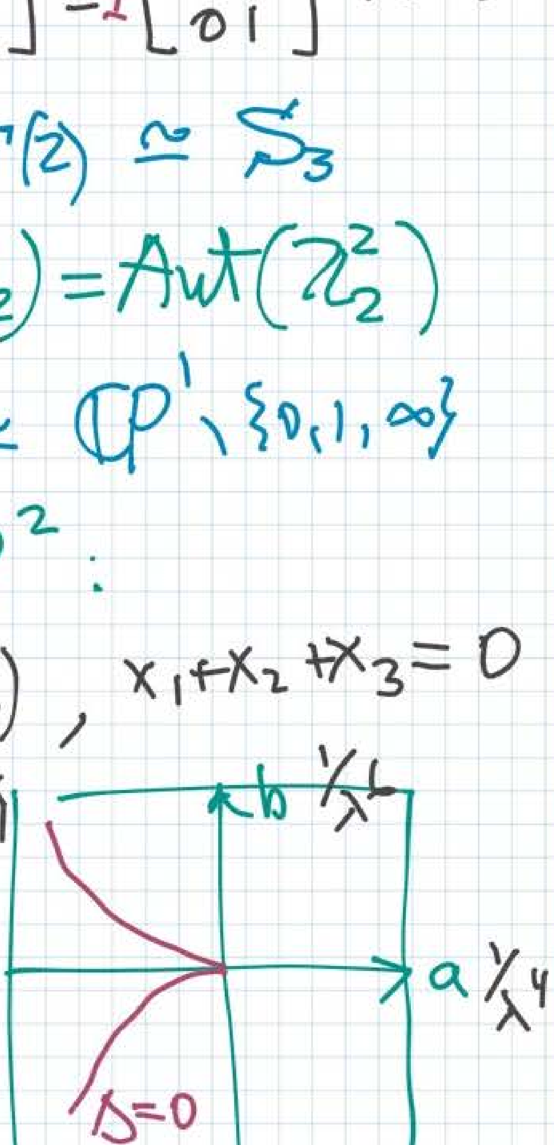
$$y^2 = (x - x_1)(x - x_2)(x - x_3), \quad x_1 + x_2 + x_3 = 0$$

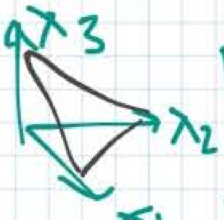


$a = x_1x_2 + x_2x_3 + x_3x_1$

$b = -x_1x_2x_3$

$\longrightarrow$ Vieta map





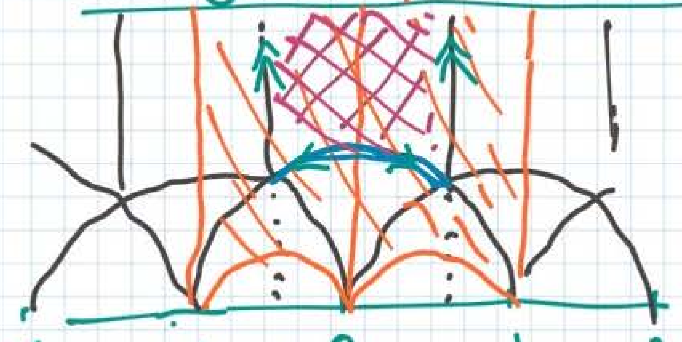
$x_1 + x_2 + x_3 = 1$

$\Omega \mapsto \lambda\,\Omega$

$$\frac{1}{\lambda^2}\begin{cases} x_1 = \wp(\omega_1/2) \\ x_2 = \wp(\omega_2/2) \\ x_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right) \end{cases}$$

1-dim subspace in $\mathbb{C}^2$

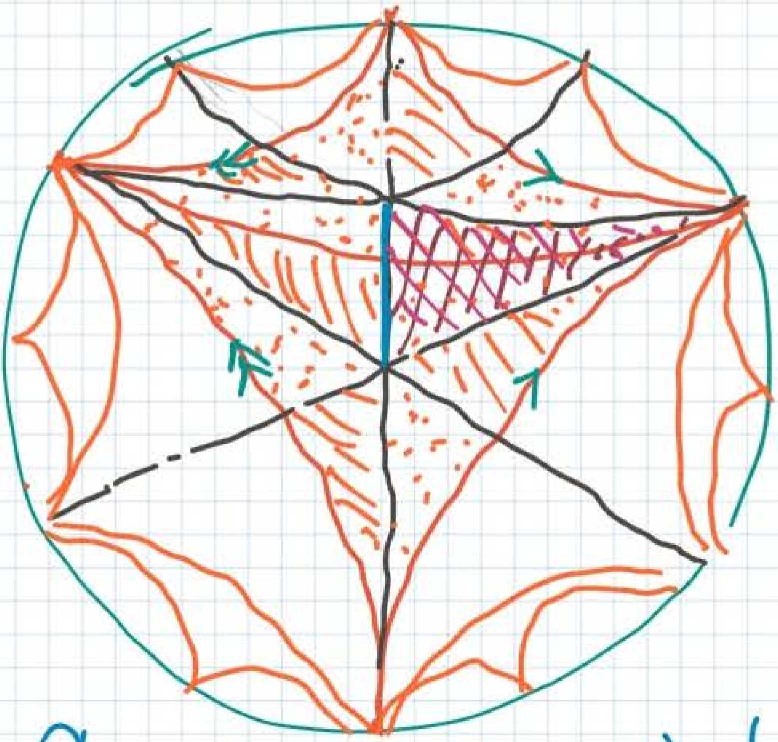
Using the modular figure:



$PSL_2(\mathbb{Z})/\Gamma(2) \simeq S_3$

$S^2 = I = T^2$

$(ST)^3 = I$



Corollary

$\Gamma(2)$ acts on $\mathbb{H} \simeq \mathbb{U}$ without fixed points.

Corollary (Picard's Little Thm)

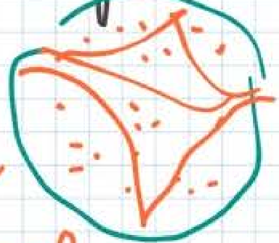
A non-constant entire function assumes all but at most one complex value.

Proof:

$$\begin{tikzcd} & \mathbb{U} \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"', "\text{suppose}"] & \mathbb{CP}^1 \setminus \{0,1,\infty\} = \mathbb{U}/\Gamma(2) \end{tikzcd}$$

simply connected

local isomorphism

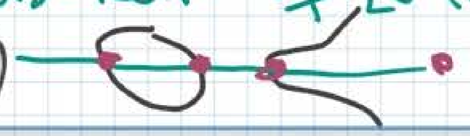


$\tilde{f}$ = single-valued branch of $\pi^{-1} \circ f$ is bounded, hence constant by Liouville's theorem.

Remark: $\left(\mathbb{C}/\Lambda, \frac{1}{2}\Lambda/\Lambda\right) \mapsto \lambda \in \mathbb{CP}^1 \setminus \{0,1,\infty\}$

λ = cross-ratio of $\left[\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1+\omega_2}{2}\right), \wp(0)\right]$

$y^2 = x(x-1)(x-\lambda)$



$[x_1, x_2, x_3, \infty]$

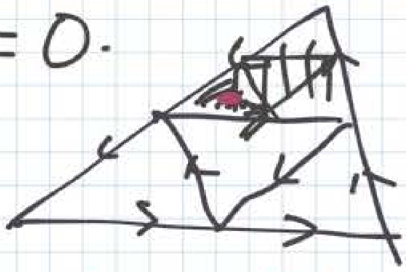
Complex Analysis. Review

Def. A mapping $z = x+iy \mapsto f = u+iv$ is called differentiable in the complex sense if its linearization $\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} \mapsto \begin{bmatrix} u_x & u_y \\ v_x & v_y\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$ is well-defined and is a multiplication by a complex number: $\begin{bmatrix} u_x & u_y \\ v_x & v_y\end{bmatrix} = \begin{bmatrix} a & -b \\ b & a\end{bmatrix}$

$\Longleftrightarrow$ Cauchy-Riemann eqns: $\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases}$

$\Longleftrightarrow$ $\dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right) = 0.$

Theorem (Cauchy) If f is holomorphic, then $\int f(z)\,dz = 0.$



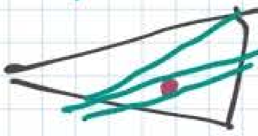
Δ_0

$|I_0| \geq \alpha > 0,\ |I_1| \geq \frac{\alpha}{4}, \dots, |I_n| \geq \frac{\alpha}{4^n}.$

$f(z) = f(z^*) + f'(z^*)\Delta z + o(|\Delta z|)$

$\Rightarrow \left|\int_{\Delta_n} f(z)\,dz\right| = o\left((\operatorname{diam}\Delta_n)^2\right)$ contradiction

Remark: Remains true even if at one point f is only continuous.



$>\varepsilon \to 0$

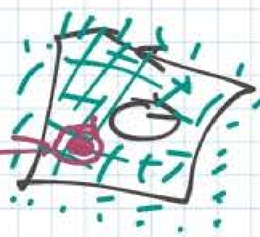
Corollary $\underset{\text{holom.}}{f(z)}\,dz \underset{\text{locally}}{\Longrightarrow} d\,g(z)$ ← holom. $\left(\frac{\partial g}{\partial \bar z} = 0\right)$

Corollary (Cauchy's formula)

$$f(z) = \frac{1}{2\pi i}\oint_{|t|=r} \frac{f(t)\,dt}{t-z} \qquad \frac{f(t)-f(z)}{t-z}\,dt = d\,g(t)$$

Corollary: f is infinitely differentiable

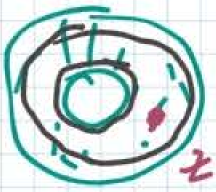
Corollary: $f(z) = \dfrac{1}{2\pi i}\displaystyle\oint_{\partial K} \frac{f(t)\,dt}{t-z}$



z t

Proof: Green's formula for $K - \odot$

Special case: K = annulus (Laurent series). $\boxed{40.2}$



$$f(z) = \frac{1}{2\pi i}\oint_{|t|=r_2} \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\oint_{|t|=r_1} \frac{f(t)\,dt}{t-z} \qquad r_2 > r_1$$

$$= \sum_{n\in\mathbb{Z}} a_n z^n \qquad a_n = \frac{1}{2\pi i}\oint_{|t|=r} \frac{f(t)\,dt}{t^{n+1}}$$

Corollaries

① Cauchy's inequalities: $|a_n| \le \max_{|t|=r} |f(t)| / r^n$

② f - holom. in $|z|<\rho \Rightarrow f(z) = \sum_{n\ge 0} a_n z^n$ with conv. radius $\ge \rho$.

③ Liouville Thm:
$f: \mathbb{C} \to \mathbb{D}$ (bounded, entire) $\Rightarrow f = \text{const}$

④ Elimination of singularities:
f - bounded in a punctured nbhd of z_0
$\Rightarrow f$ is holom. at z_0.

⑤ Weierstrass' Thm: In a nbhd of an essential singularity, f takes on a dense set of values. $|f(z)-a| \ge \varepsilon > 0 \Rightarrow \left|\frac{1}{f(z)-a}\right| \le \frac{1}{\varepsilon}$
f is meromorphic at z_0 $\Leftarrow$ $\leftarrow$ holom at z_0.

⑥ MVP $a_0 = \frac{1}{2\pi}\int f(re^{i\theta})\,d\theta \Rightarrow$ MMP

⑦ Schwarz' Lemma $f: \mathbb{D},0 \to \mathbb{D},0$
$\Rightarrow |f(z)| \le |z|$, "=" $\Rightarrow f(z) = e^{i\theta} z$.

⑧ $\mathrm{Aut}\,\mathbb{D} \simeq \mathrm{Aut}\,\mathbb{H} = PSL_2(\mathbb{R})$

Remark: $\mathrm{Aut}\,\mathbb{C}P^1 = PSL_2(\mathbb{C})$ $\Leftarrow$

⑤ + Fund. Th. of Algebra $\Leftarrow$

$$\#Z - \#P = \frac{1}{2\pi i}\oint_{\partial K} \frac{df}{f} \Leftarrow \boxed{\oint_{\partial K} f(z)\,dz = 2\pi i \sum \mathrm{Res} \quad \text{residue theorem}}$$

$\Downarrow$ application to definite integrals.

⑨ $\{f^{(n)}\}$-bounded
$\Rightarrow$ equicontinuous (*)
$\Rightarrow$ contains a convergent subsequence

(*) - Series of meromorphic functions,
- Infinite products
- $\wp$-functions
- Elliptic curves
- Γ-function.

Theorem (Riemann)

A simply connected $D \subsetneq \mathbb{C}$ is isomorphic to $\mathbb{D}$

Picard's Little Thm $\Leftarrow$