Chapter III. An Equivariant Cohomology Theory Related to Fibre Bundle Theory

In the application of cohomology theory to the study of topological transformation groups, a natural and convenient formalism is to define an *equivariant cohomology theory* for the category of G-spaces which *effectively* reflects the cohomological behavior of both *the space* and *the G-action*. Following an idea of A. Borel [cf. B 10], we shall define the *equivariant cohomology* of a G-space X to be the *ordinary cohomology* of the *total space* X_G of the *universal* bundle, $X \to X_G \to B_G$, with the given G-space X as its typical fibre, namely

$$H_G^*(X) \overset{\text{def}}{=\!=} H^*(X_G), \quad \text{where} \quad X_G = E_G \times_G X = (E_G \times X)/G.$$

The rationale of adopting the above equivariant cohomology theory in terms of the universal bundle construction is roughly the following:

(i) Intuitively and heuristically, the complexity of the G-action on X will be reflected in the complexity of the associated universal bundle $X \to X_G \to B_G$, e.g., the associated universal bundle is trivial if and only if the G-action on X is trivial. And the classical obstruction theory, especially the characteristic classes theory, clearly demonstrates that cohomology theory can then be used to detect the complexity of $X_G \to B_G$, which, in turn, reflects the complexity of the G-action itself.

(ii) Technically, it is not difficult to see that such an equivariant cohomology theory not only possesses convenient formal properties but is also effectively computable.

§ 1. The Construction of $H_G^*(X)$ and its Formal Properties

(A) *The Construction of A. Borel*

Let X be a given G-space and $E_G \to B_G$ be the universal G-bundle. Then the total space X_G of the associated universal bundle with X as fibre may be regarded as: the orbit space of $E_G \times X$

$$X_G = E_G \times_G X = (E_G \times X)/G$$

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where the G-action is given by $g\cdot(e,x)=(eg^{-1},gx)$. Since the two projections are obviously equivariant, one has the following commutative diagram:

$$\begin{tikzcd}
E_G \arrow[d] & E_G\times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
B_G & X_G \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & X/G
\end{tikzcd}$$

Next suppose that Y is a K-space, $h\colon G\to K$ is a homomorphism and $f\colon X\to Y$ is an h-equivariant map, i.e., $f(g\cdot x)=h(g)\cdot f(x)$. Then, it is easy to check that, correspondingly, there is the following commutative diagram:

$$\begin{tikzcd}
E_G \arrow[dd] \arrow[dr, "E_h"] & & E_G\times X \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dr, "E_h\times f"] & & X \arrow[dd] \arrow[dr, "f"] & \\
& E_K & & E_K\times Y \arrow[ll, crossing over] \arrow[rr, crossing over] & & Y \arrow[dd] \\
B_G \arrow[dr, "B_h"] & & X_G \arrow[ll] \arrow[rr] \arrow[dr, "f_h"] & & X/G \arrow[dr, "\bar f"] & \\
& B_K \arrow[from=uu, crossing over] & & Y_K \arrow[ll] \arrow[rr] \arrow[from=uu, crossing over] & & Y/K
\end{tikzcd}$$

Hence, it is clear that the above construction is functorial and it follows readily that its composition with the ordinary cohomology theory will yield an *equivariant cohomology theory.*

Definition. Let $\mathscr{E}$ be the equivariant category of spaces with topological actions and equivariant maps. Then the following functor

$$(G\text{-space } X) \longmapsto H_G^*(X)=H^*(X_G),$$
$$(h\text{-equivariant map}: X \xrightarrow{f} Y) \longmapsto f_h^*: H_K^*(Y) \longrightarrow H_G^*(X)$$

is called the equivariant cohomology functor.

(B) *The Coefficient System of $H_G^*(\cdot)$*

For a *fixed* group G, it is obvious that all G-spaces and G-equivariant maps form a sub-category $\mathscr{E}_G$, and the restriction of the above theory to $\mathscr{E}_G$ will be simply called the H_G^*-theory. Since the set of homogeneous G-spaces, $\{G/H,\ H$ closed subgroups of $G\}$, are exactly those atomic G-spaces, their values of H_G^*-functor can be considered as the *coefficient system* of H_G^*-theory. Observe that

$$(G/H)_G=E_G\times_G(G/H)=(E_G\times_G G)/H=(E_G)/H=B_H\,,$$

and equivariant maps between homogeneous G-spaces are given by

$$G/H \to G/K \quad \text{for pairs of subgroups} \quad H \subseteq K.$$

Therefore, the coefficient system of the H_G^*-theory consists of the following algebras and morphisms:

$$H_G^*(G/H) = H^*(B_H) \quad \text{for each closed subgroup } H \subseteq G,$$
$$H_G^*(G/K) = H^*(B_K) \to H^*(B_H) = H_G^*(G/H) \quad \text{for each pair } H \subseteq K,$$

where the above morphism is induced by $B_H = E_G/H \to E_G/K = B_K$.

Examples. As an important example for later development, we shall compute the coefficient system of rational H_G^*-theory for compact Lie groups G.

Reduction 1. Let G be a compact Lie group and G^0 be the identity component of G. Then $\Gamma = G/G^0$ is a *finite* group and

$$B_{G^0} = E_G/G_0 \xrightarrow{p} E_G/G = B_G$$

is a covering space with $\Gamma = G/G^0$ acting as deck transformations.

Hence, it follows from a simple theorem of Grothendieck [G 5] that $p^*: H^*(B_G; \mathbb{Q}) \cong H^*(B_{G^0}; \mathbb{Q})^\Gamma$ (fixed elements under the induced action of Γ).

Lemma (1.1). *Let G be a compact connected Lie group, T be a maximal torus, $N(T)$ be the normalizor of T in G, $W = N(T)/T$ be the Weyl group of G. Then $H^*(G/N(T); \mathbb{Q}) \cong H^*(G/T; \mathbb{Q})^W \cong H^*(pt; \mathbb{Q})$, i.e., $G/N(T) \sim_{\mathbb{Q}} pt$.*

Proof. Since $W \to G/T \to G/N(T)$ is a finite covering, it follows that

$$H^*(G/N(T); \mathbb{Q}) \cong H^*(G/T; \mathbb{Q})^W \quad \text{and} \quad \chi(G/N(T)) = \frac{1}{|W|} \cdot \chi(G/T).$$

On the other hand, the well-known Bruhat decomposition induces a cell-decomposition of G/T with exactly $|W|$ cells of even dimension [p. 347, B 7]. Hence

$$H^{\text{odd}}(G/T; \mathbb{Q}) = 0 \quad \text{and} \quad \dim_{\mathbb{Q}} H^*(G/T; \mathbb{Q}) = \chi(G/T) = |W|$$

and consequently,

$$H^{\text{odd}}(G/N(T); \mathbb{Q}) \cong H^{\text{odd}}(G/T; \mathbb{Q})^W = 0,$$
$$\dim_{\mathbb{Q}} H^*(G/N(T); \mathbb{Q}) = \chi(G/N(T)) = \frac{1}{|W|} \cdot \chi(G/T) = 1, \text{ i.e., } G/N(T) \sim_{\mathbb{Q}} pt. \quad \square$$

Reduction 2. Let G be a compact connected Lie group, T be a maximal torus and W be the Weyl group acting as an automorphism group of T. Then

$$H^*(B_G; \mathbb{Q}) \cong H^*(B_{N(T)}; \mathbb{Q}) \cong H^*(B_T; \mathbb{Q})^W.$$

Proof. Since the fibre of the bundle $G/N(T) \longrightarrow B_{N(T)} \xrightarrow{\pi} B_G$ is $\mathbb{Q}$-acyclic, it follows easily from Serre spectral sequence that $\pi^*: H^*(B_G; \mathbb{Q}) \to H^*(B_{N(T)}; \mathbb{Q})$ is an isomorphism. Hence, one has

$$H^*(B_G; \mathbb{Q}) \cong H^*(B_{N(T)}; \mathbb{Q}) \cong H^*(B_T; \mathbb{Q})^W. \quad \square$$

Example 1. $G = U(n)$, Then $T^n = \{\mathrm{diag}(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})\}$ is a maximal torus and W acts on T by permuting θ's; and $H^*(B_{T^n}; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]$ where $\{x_1, \ldots, x_n\}$ are respectively the transgression of the basis of $H^1(T^n; \mathbb{Q})$ corresponding to $\{\theta_1, \ldots, \theta_n\}$. Hence W acts on $H^*(B_{T^n}; \mathbb{Q})$ as permutations of the x's and

$$H^*(B_G; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]^W \cong \mathbb{Q}[c_1, c_2, \ldots, c_n]$$

is exactly the ring of symmetric polynomials and the universal Chern classes $c_1, \ldots, c_n$ are respectively the elementary symmetric polynomials.

Example 2. $G = \mathrm{SO}(2n+1)$. Then

$$T^n = \left\{ \begin{pmatrix} \begin{matrix} \cos{}_1 & -\sin\theta_1 \\ \sin{}_1 & \cos\theta_1 \end{matrix} & & & 0 \\ & \ddots & & \\ & & \begin{matrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{matrix} & \\ 0 & & & 1 \end{pmatrix} \right\}$$

is a maximal torus and W acts on T^n by permuting θ's and changing signs. Hence

$$H^*(B_{\mathrm{SO}(2n+1)}; \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]^W \cong \mathbb{Q}[p_1, p_2, \ldots, p_n]$$

is the ring of symmetric polynomials in x_j^2, where the universal Pontrjgin classes $p_1, p_2, \ldots, p_n$ are respectively the elementary symmetric polynomials in x_j^2.

Example 2'. $G = \mathrm{SO}(2n)$. Then W acts on T^n by permuting θ's and changing even number of signs. Hence $e = x_1 \cdot x_2 \cdots x_n$ is also fixed under W and

$$H^*(B_{\mathrm{SO}(2n)}; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \ldots, p_{n-1}, e];\ e^2 = p_n, e \text{ is the universal Euler class.}$$

(C) *Spectral Sequences Related to the Equivariant Cohomology Theory*

In the above construction of Borel, the space X_G is constructed together with two canonical projections, namely, $\pi_1: X_G \to B_G$ and $\pi_2: X_G \to X/G$. Therefore, in the framework of cohomology theory, there are the *Serre spectral sequence* of the fibre map π_1 and the *Leray spectral sequence* of π_2 that offer useful ways to study the equivariant cohomology $H_G^*(X) = H^*(X_G)$.

Serre spectral sequence of π_1. For a given fibration: $X \xrightarrow{i} M \xrightarrow{\pi} B$ over a cell complex B, the skeleton filtration, $\{B^p = p\text{-skeleton of } B\}$, of the base space B lifts to a filtration, $\{M^p = \pi^{-1}(B^p)\}$, of the total space M. Following the usual procedure of constructing a spectral sequence from a space with a given filtration, one obtains the Serre spectral sequence of the fibration which is the main tool for analyzing the cohomological (or homological) relationship between fibre, base and the total space. The following are some of the basic facts which are useful in explicit computations of $H_G^*(X)$. (We refer to [E 3, M 1, S 3] for a thorough discussion of spectral sequences.)

The Serre spectral sequence consists of a sequence of *bigraded differential* algebras $\{(E_n^{p,q}, d_n); n \geqslant 1\}$ such that

(i) $d_n: E_n^{p,q} \to E_n^{p+n,q-n+1}$ has bigrade $(n, -n+1)$, $d_n^2 = 0$ and the homology of $(E_n^{p,q}, d_n)$ is exactly $(E_{n+1}^{p,q})$.

(ii) $E_1^{p,q} = C^p(B, H^q(X))$ and $E_2^{p,q} = H^p(B, \underline{H^q(X)})$ where $\underline{H^q(X)}$ is the local system of cohomology of fibres.

(ii)′ In the special case of $X_G \xrightarrow{\pi_1} B_G$ with connected G and rational coefficients, then it follows from the simply connectedness of B_G and Kunneth formula that $E_2^{p,q} = H^p(B, \mathbb{Q}) \otimes H^q(X; \mathbb{Q})$.

(iii) $E_n^{p,q} = 0$ for $p < 0$ or $q < 0$, and $E_n^{p,q} = E_{n+1}^{p,q}$ for $n > (p+q+1)$. Hence $E_\infty^{p,q} = E_n^{p,q}$ for $n > (p+q+1)$ is well defined. Moreover, $(E_\infty^{p,q})$ is the associated graded algebra of $H^*(M)$ w.r.t. the filtration

$$F^p H^*(M) = \ker\{H^*(M) \to H^*(M^{p-1})\},$$

namely

$$E_\infty^{p,q} = F^p H^{p+q}(M) / F^{p+1} H^{p+q}(M)$$

(iv) The following two edge homomorphisms are respectively the induced homomorphism i^* and π^*, namely

$$i^*: H^*(M) \to E_\infty^{0,*} \subseteq H^*(X);\ E_\infty^{0,*} = Im(i^*),$$

$$\pi^* H^*(B) \to E^{*,0} \subseteq H^*(M);\ E^{*,0} = Im(\pi^*).$$

Leray spectral sequence of π_2. The Leray spectral sequence [B 10] of a map $\pi: Y \to Z$ is a spectral sequence $\{E_n, d_n\}$, i.e., $E_{n+1} = H(E_n, d_n)$, which begins with $E_2 = H^*(Z; \mathscr{S})$ and converges to $H^*(Y)$, where $\mathscr{S}$ is the *coefficient sheaf* over Z with $H^*(\pi^{-1}(z))$ as its stalk over $z \in Z$. In the case of $\pi_2: X_G \to X/G$, the inverse image $\pi^{-1}(x')$ of $x' = G(x) \in X/G$ can easily be computed as follows:

$$\pi_2^{-1}(x') = E_G \times_G (G/G_x) = (E_G \times_G G)/G_x = E_G/G_x = B_{G_x}.$$

Hence the E_2-term of Leray spectral sequence of π_2 is equal to $H^*(X/G, \mathscr{S})$, where the stalk of $\mathscr{S}$ over x' is $H^*(B_{G_x})$. In general, it is extremely difficult to compute the Leray spectral sequence beyond the E_2-term. However, the above E_2-term provides a precise description of how the coefficient system of H_G^*-theory plays its rôle, and consequently, the above knowledge of E_2-term alone turns out to be quite useful.

Next, let us consider the following problem.

Problem. Let X be a given G-*space* and K be a closed subgroup of G. Then the restriction of G-action to K makes X into a K-space. What is the relationship between $H_G^*(X)$ *and* $H_K^*(X)$?

Clearly, one may take $E_K = E_G$ with the restricted K-action. Then, one has the following commutative diagram of fibrations

$$\begin{array}{ccccc} & & X & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ G/K & \longrightarrow & X_K & \longrightarrow & X_G \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \longrightarrow & B_K & \longrightarrow & B_G . \end{array}$$

Exactly for such a geometric situation, Eilenberg-Moore [E3] constructed a spectral sequence $\{E_n, d_n\}$ such that

$$E_n \Rightarrow H^*(X_K) = H^*{}_K(X),$$
$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H^*(B_G)}(H^*(B_K), H^*(X_G)).$$

The following are some simple important cases that deserve special attention:

Example 1, $K = \{\mathrm{id}\}$. Then the above spectral sequence reduces to

$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H^*(B_G)}(H^*(pt), H^*(X_G)),\ E_n \Rightarrow H^*(X).$$

Example 2, Let X, Y be two G-spaces, Then $X \times Y$ is a $(G \times G)$-space and its restriction to the diagonal subgroup $G \xrightarrow{\Delta} (G \times G)$ makes $X \times Y$ into a G-space. Hence, one has a spectral sequence with

$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H^*(B_G \times B_G)}(H^*(B_G), H^*(X_G \times Y_G))$$

and

$$E_n \Rightarrow H_G^*(X \times Y).$$

This is the Kunneth spectral sequence of H_G^*-theory.

Example 3, $K = G^0$ (the identity component of G); $\Gamma = G/G^0$ is finite. Then it follows easily from the fibration $\Gamma \to X_{G^0} \to X_G$ that

$$H_G^*(X; \mathbb{Q}) \cong H_{G^0}^*(X; \mathbb{Q})^\Gamma.$$

Proposition 1. *Let G be a compact connected Lie group, T be a maximal torus of G, $W = N(T)/T$ be the Weyl group of G and X be a G-space. Then*

(i) $$H_G^*(X; \mathbb{Q}) \cong H_{N(T)}^*(X; \mathbb{Q}) \cong H_T^*(X; \mathbb{Q})^W,$$

(ii) $$H_T^*(X; \mathbb{Q}) \cong H_G^*(X; \mathbb{Q}) \otimes_{H_G^*(pt)} H_T^*(pt)$$
$$= H_G^*(X; \mathbb{Q}) \otimes_{H^*(B_G, \mathbb{Q})} H^*(B_T, \mathbb{Q}).$$

Proof. (i) follows from the $\mathbb{Q}$-acyclicity of $G/N(T)$ and the Serre spectral sequence of the fibration $G/N(T) \to X_{N(T)} \to X_G$.

(ii) follows from the above Eilenberg-Moore spectral sequence and the fact that $H^*(B_T, \mathbb{Q}) \cong H^*(B_G; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(G/T; \mathbb{Q})$ is a *free-$H^*(B_G; \mathbb{Q})$ module.* Thus, the E_2-term reduces to one line;

$$E_2^{p,q} = 0 \quad \text{for} \quad q \neq 0 \quad \text{and}$$

$$E_2^{*,0} = \mathrm{Tor}^{*,0}_{H^*(B_G, \mathbb{Q})}(H^*(B_T; \mathbb{Q}), H_G^*(X; \mathbb{Q})) = H_G^*(X; \mathbb{Q}) \otimes_{H^*(B_G;\, B)} H^*(B_T; \mathbb{Q}). \quad \square$$

Remark. In view of the maximal torus theorem and the unique central rôle of torus groups in the cohomology theory of transformation groups, the above simple neat result is in fact of basic importance.

§ 2. Localization Theorem of Borel-Atiyah-Segal Type

Let $R = H^*(B_G) = H_G^*(pt)$. Then it is clear that all $H_G^*(X, Y)$ are modules over R and all induced H_G^*-morphisms are morphisms of R-modules. Recall that an element m of an R-module M is said to be (R-)torsion if there exists $r \neq 0 \in R$ such that $r \cdot m = 0$.

Historically, the following simple fact plays a crucial rôle in Borel's approach to the cohomology theory of transformation groups [B 10].

Proposition 2 (A. Borel). *Let G be the circle group, X be a finite dimensional G-space, $F = F(G, X)$ be the fixed point set of X. Then*

(i) *$H_G^*(X - F; \mathbb{Q}) = H^*((X - F)/G; \mathbb{Q})$ is a torsion R-module.*

(ii) *the* Ker *and* Coker *of* $H_G^*(X, \mathbb{Q}) \xrightarrow{r^*} H_G^*(F; \mathbb{Q}) = R \otimes H^*(F; \mathbb{Q})$ *are both torsion R-modules.*

Proof. Since G_x are finite groups for all $x \in (X - F)$, $H^*(B_{G_x}; \mathbb{Q}) = \mathbb{Q}$ for all $x \in (X - F)$. Therefore, the E_2-term of the Leray spectral sequence of $\pi_2 : (X - F)_G \to (X - F)/G$ reduces to one line, namely

$$E_2^{p,q} = 0 \quad \text{if} \quad q \neq 0 \quad \text{and} \quad E_2^{p,0} = H^p((X - F)/G; \mathbb{Q}).$$

Hence $H_G^*(X - F; \mathbb{Q}) \cong H^*((X - F)/G; \mathbb{Q})$ which is finite dimensional and obviously R-torsion. Clearly. (ii) follws directly from (i) and the exact sequence of the pair (X, F) in H_G^*-theory. $\square$

Algebraically, the above proposition can easily be reformulated into the following formally neater statement, namely,

$$H_G^*(X; \mathbb{Q}) \otimes_R \hat{R} \xrightarrow{i^* \otimes \hat{R}} H_G^*(F; \mathbb{Q}) \otimes_R \hat{R} = H^*(F; \mathbb{Q}) \otimes_{\mathbb{Q}} \hat{R}$$

is an isomorphism, where $\hat{R}$ is the quotient field of R. This is the primitive version of localization theorems of Atiyah-Segal type [A 8]. Recall that a multiplicative semigroup, $S \supseteq \{1\}$, contained in the center of R is called a *multiplicative system*; and the localized module $S^{-1}M$ of an R-module M consists of all fractions

$\{m/s, m\in M, s\in S\}$ with the usual identification $m_1/s_1 = m_2/s_2$ if and only if $ss_1 m_2 = ss_2 m_1$ for some $s\in S$. Then $S^{-1}M$ is an $(S^{-1}R)$-module and $M\mapsto S^{-1}M$ is an *exact functor*. Now, let us consider the case $R = H^*(B_G) = H_G^*(pt)$ and $M = H_G^*(X, Y)$. For a given multiplicative system $S\subseteq R$, we set

$$X^S = \{x\in X \mid \text{no element of } S \text{ maps to zero in } R\to H^*(B_{G_x})\}.$$

Theorem (III.1) (Localization). *Let G be a compact Lie group and X be a compact G-space. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X)\to S^{-1}H_G^*(X^S)$$

is an isomorphism.

Proof. (i) We shall first prove the special case $X^S = \emptyset$. In this case, we need only to show that there exists $s\in S$ such that $\pi_1^*(s) = 0$ in $H_G^*(X)$. By the slice theorem [cf. Th. (I.5), §3, Ch. I], each orbit $G(x)$ has invariant open neighborhood U such that $G(x)$ is an equivariant retract of U. By compactness of X, there are finite number of such neighbornhoods $\{U_1, \dots, U_q\}$ covering X, with U_i retracting to $G(x_i)$. Since X^S is assumed to be empty, there is an $s_i\in S$ which maps to zero in $H^*(B_{G_{x_i}}) = H_G^*(G(x_i))$ and hence also in $H_G^*(U_i)$. Then it is clear that $\pi_1^*(s_1\cdot \cdots \cdot s_q) = 0$ in $H_G^*(U_1\cup\cdots\cup U_q) = H_G^*(X)$.

(ii) The general case is equivalent to showing $S^{-1}H_G^*(X, X^S) = 0$ which means that for every $x\in H_G^*(X, X^S)$ there is an $s\in S$ with $s\cdot x = 0$. We may assume that

$$x\in H_G^n(X, X^S) = H^n(X_G, X_G^S) = H^n(\pi^{-1}(B_G^k), X_G^S\cap\pi^{-1}(B_G^k) \quad \text{for} \quad k>n.$$

Since $\pi^{-1}(B_G^k)$ is compact and the neighborhood of $X_G^S\cap\pi^{-1}(B_G^k)$ in $\pi^{-1}(B_G^k)$ of the form $\{V_G\cap\pi^{-1}(B_G^k);\ V \text{ invariant neighborhood of } X^S\}$ are cofinal, it follows from the continuity of Čech cohomology that there exists an invariant neighborhood V of X^S such that

$$x\in \operatorname{Im}\{H_G^*(X, V)\to H_G^*(X, X^S)\}.$$

On the other hand, there exist an invariant compact subspace $Y\subseteq (X - X^S)$ such that $V\cup \operatorname{int}(Y) = X$. Hence, it follows from (i) and the fact $Y^S = \emptyset$ that there is $s\in S$ with $\pi_1^*(s)\in \operatorname{Im}\{H_G^*(X, Y)\to H_G^*(X, \operatorname{int} Y)\to H_G^*(X)\}$. Therefore $s\cdot x$ lies in the image of

$$0 = H_G^*(X; V\cup \operatorname{int} Y)\to H_G^*(X, X^S), \quad \text{i.e.,} \quad s\cdot x = 0. \quad \square$$

Theorem (III.1′). *Let G be a compact Lie group and X be a G-space, paracompact and with finite cohomological dimension. Let $S\subseteq H^*(B_G)$ be a multiplicative system and $s\in S$. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X)\to S^{-1}H_G^*(X^s)$$

is an isomorphism. If X consists of only finite orbit types, then

$$S^{-1} H_G^*(X) \to S^{-1} H_G^*(X^S)$$

is also an isomorphism.

Proof. We shall first prove the case $X^s = \emptyset$. Since X is assumed to be of finite cohomological dimension, it is not difficult to show that the cohomological dimension $\mathrm{cd}(X/G) \leqslant \mathrm{cd}(X)$ is also finite [see Q1]. Hence the E_2-term of Leray spectral sequence of $\pi_2 : X_G \to X/G$ is bounded from the right, i.e., $E_2^{p,q} = 0$ for p bigger than a fixed, sufficiently large N. Therefore,

$$E_\infty^{p,q} = 0 \quad \text{for} \quad p > N$$

and there is a decreasing filtration $F^p H_G^*(X)$ with $F^{N+1} H_G^*(X) = 0$ satisfying:

(i) $a \in F^p$, $b \in F^{p'}$ imply $a \cdot b \in F^{p+p'}$,

(ii) $$E_\infty^{p,*} = F^p H_G^*(X)/F^{p+1} H_G^*(X).$$

The assumption $X^s = \emptyset$ simply means s maps to zero in every stalk $\mathscr{S}_{x'} = H^*(B_{G_x})$. Hence s maps to zero in

$$E_2^{0*} = H^0(X/G; \mathscr{S}) \quad \text{and also in} \quad E_\infty^{0,*}.$$

Therefore, it follows from the following exact sequence

$$0 \to F^1 \to F^0 = H_G^*(X) \to E_\infty^{0,*} \to 0$$

that $\pi_1^*(s) \in F^1 H_G^*(X)$. Then, it follows easily that

$$\pi_1^*(s^{N+1}) \in F^{N+1} H_G^*(X) = 0, \quad \text{i.e.,} \quad \pi_1^*(s^{n+1}) = 0$$

which clearly implies $S^{-1} H_G^*(X) = 0$ (for $s^{N+1} \in S$).

The transition from the case $X^s = \emptyset$ to the case $X^s \neq \emptyset$ is the same as in the compact case. We shall show that

$$S^{-1} H_G^*(X) \to S^{-1} H_G^*(X^S)$$

is also an isomorphism under the assumption of finite orbit types. Let $G_1, G_2, \ldots, G_n$ be the orbit types in $X - X^S$ and $s_i \in S$ maps to zero in $H^*(B_{G_i})$. Then it is clear that

$$X^S = X^s \quad \text{for} \quad s = s_1 \cdot s_2 \cdot \cdots \cdot s_n$$

and hence the above proof applies. ▯

Remark. (i) The above proof in fact shows that

$$s^{N+1} H_G^*(X, X^s) = 0 \quad \text{for} \quad N \geqslant \mathrm{cd}(X),$$

which is sometimes a useful fact.

(ii) Heuristically, both the statement and the proof of the above localization theorem can be obtained by applying the localization functor S^{-1} to the Leray spectral sequence of π_2. Since the localization functor S^{-1} is exact, it is reasonable to expect that

$$S^{-1}E_2 = H^*(X/G; S^{-1}\mathscr{S}), (S^{-1}\mathscr{S})_{x'} = S^{-1}(\mathscr{S}_{x'}) = S^{-1}H^*(B_{G_x})$$

and $\{S^{-1}E_n\} \Rightarrow S^{-1}H_G^*(X)$.

Hence, $S^{-1}\mathscr{S}_{x'} = 0$ for $x \in X - X^S$ and it follows easily that $S^{-1}H_G^*(X, X^S) = 0$.