Lecture 25 | Equivariant Cohomology

Let a compact topological group G act on a topological space X (a G-space)

Put $X_G := (X \times EG)/G$ - **homotopy quotient**

Motivation: $X \times EG \sim X$, but the action is free

Define $H^\bullet_G(X) := H^\bullet(X_G)$

Name: G-equivariant cohomology of X.

Examples: 1° If the action is free, then

$$X_G = (X/G) \times EG \Rightarrow H^\bullet_G(X) = H^\bullet(X/G)$$

2° $X = pt \Rightarrow X_G = BG \Rightarrow H^\bullet_G(pt) = H^\bullet(BG)$

A G-equivariant map $X \xrightarrow{f} Y$ of G-spaces induces $H^\bullet_G(Y) \xrightarrow{f^*} H^\bullet_G(X)$ $\left(X_G \xrightarrow{f_G} Y_G\right)$

E.g. $X \to pt \Rightarrow H^\bullet_G(X)$ is a $H^\bullet_G(pt)$-module

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & pt \end{tikzcd} \Rightarrow f^* \text{ is a morphism of } H^\bullet_G(pt)\text{-modules}$$

Example: If the action is free, then

$$\begin{tikzcd} X \times EG \arrow[r] \arrow[d, "G"'] & pt \times EG \arrow[d, "G"] \\ (X/G) \times EG \arrow[r] & BG \end{tikzcd}$$

principal G-bundle; inducing

$\Rightarrow$ The $H^\bullet_G(pt)$-module structure on $H^\bullet(X/G)$ is via the characteristic classes of the G-bundle.

$G' \xrightarrow{h} G$ - group homomorphism $\Rightarrow$

$EG' \to EG$ (e.g. Milnor's construction)

$$\Rightarrow X_{G'} \to X_G \Rightarrow h^*: H^\bullet_G(X) \to H^\bullet_{G'}(X)$$

E.g. $H < G \Rightarrow (X \times EG)/H = X_H \to BH$

$$\begin{tikzcd} X_H \arrow[r] \arrow[d, "G/H"] & BH \arrow[d, "G/H"] \\ X_G \arrow[r] & BG \end{tikzcd}$$

Related by spectral sequence

Example: $T^n < U_n \Rightarrow$

$$\begin{tikzcd} X_{T^n} \arrow[d, "U_n/T^n"] \\ X_{U_n} \end{tikzcd}$$

fiber = mfld of complete flags

L.25.2) Two spectral sequences

$$X_G = (X \times EG)/G \qquad \text{Serre's spectral sequence}$$

$$\begin{array}{l} X_G \\ \downarrow X \\ BG = EG/G \end{array} \qquad E_2^{\bullet,\bullet} = H^\bullet(BG; H^\bullet(X)) \rightsquigarrow H^\bullet_G(X)$$

$\Longrightarrow$ G-connected, then over $\mathbb{Q}$, $E_2^{\bullet,0}$ is a free $H^\bullet_G(pt;\mathbb{Q})$-module of rank $\dim_{\mathbb{Q}} H^\bullet(X;\mathbb{Q})$

$$\Longrightarrow \boxed{\operatorname{rk}_{H^\bullet_G(pt)} H^\bullet_G(X) \le \operatorname{rk} H^\bullet(X)}$$

and "=" only when $E_2 = E_\infty$ over $\mathbb{Q}$

$$X_G = (X \times EG)/G \qquad \text{Leray's spectral sequence}$$

$$\begin{array}{l} X_G \\ \downarrow BG_x \\ X/G \ni [x] \end{array} \qquad E_2^{\bullet,\bullet} = H^\bullet(X/G; \text{"}H^\bullet_{G_x}(pt)\text{"}) \rightsquigarrow H^\bullet_G(X)$$

Key example: $G = T^n$, $H^\bullet_{T^n}(pt) = \mathbb{Z}[\lambda_1, \dots, \lambda_n]$

$\mathbb{R}^N \supset X$ is finite-dim $\overset{i}{\supseteq} X^T$ – fixed pt set

A. Borel's Localization Thm:

$$H^\bullet_T(X;\mathbb{Q}) \xrightarrow{i^*} H^\bullet_T(X^T;\mathbb{Q}) = H^\bullet(X^T) \otimes \mathbb{Q}[\lambda]$$

becomes an isomorphism after tensoring over $\mathbb{Q}[\lambda]$ with the field of fractions $\mathbb{Q}(\lambda)$.

Pf: $T^n_x \simeq T^k \times F$inite (abelian) group

$$\tilde{H}^\bullet(K(F,1);\mathbb{Q}) = 0, \quad H^\bullet(BT^k) \otimes_{\mathbb{Q}[\lambda]} \mathbb{Q}(\lambda) = 0 \text{ if } k < n$$

$\Longrightarrow$ "$H^\bullet_{T_x}(pt)$" $= 0$ outside X^T

Corollary: $\operatorname{rk}_{H^\bullet_T(pt)} H^\bullet_T(X) = \operatorname{rk} H^\bullet(X^T) \le \operatorname{rk} H^\bullet(X)$

"Smith inequality"

Variant:

$\mathbb{Q} \rightsquigarrow \mathbb{Z}_2$ (or $\mathbb{Z}_p$)

$T^n \rightsquigarrow \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2 \qquad H^\bullet_{\mathbb{Z}_2^n}(pt) = \mathbb{Z}_2[\lambda_1, \dots, \lambda_n]$

Thm: $H^\bullet_{\mathbb{Z}_2^n}(X;\mathbb{Z}_2) \xrightarrow{i^*} H^\bullet_{\mathbb{Z}_2^n}(X^T;\mathbb{Z}_2)$ – isomorphic over $\mathbb{Z}_2(\lambda)$

1.25.3) Equivariant char. classes

$$\begin{array}{ccc} E & \text{- } G\text{-equivariant} & E_G = (E \times EG)/G \\ \xi \downarrow V & \text{(say, complex)} & \xi_G \downarrow V \\ X & \text{vector bundle} & X_G = (X \times EG)/G \end{array}$$

$c_k(\xi_G) \in H^{2k}(X_G)$ - G-equivariant Chern classes of ξ

Examples 1. S^1 acts on $\mathbb{C}$ by $e^{2\pi\sqrt{-1}\,\ell t}$

$\downarrow \mathbb{C}$ $\quad \downarrow L^{\ell}$ $\quad c_1 = -\ell\lambda := c_1(L^{-1})$ hyperplane class

$\cdot\, pt \quad BS^1 = \mathbb{C}P^\infty$

2. $T^n \to S^1 : (t_1, \dots, t_n) \mapsto e^{2\pi\sqrt{-1}(\ell_1 t_1 + \dots + \ell_n t_n)}$

$\Rightarrow c_1 = -(\ell_1\lambda_1 + \dots + \ell_n\lambda_n)$ function on Lie T^n

3. $X = \mathbb{C}P^{n-1} = \mathrm{proj}(\mathbb{C}^n)$ $\quad T^n = \begin{bmatrix} e^{2\pi i t_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i t_n} \end{bmatrix}$

$x := c_1(L^{-1})$ $\quad x|_{k\text{-th fixed pt}} = \lambda_k,\ k = 1, \dots, n.$

$X_T \quad E_2 = H^\bullet(\mathbb{C}P^{n-1}) \otimes_{\mathbb{Z}} H^\bullet(BT) = E_\infty$

$\downarrow \mathbb{C}P^{n-1}$

$BT \quad n = \mathrm{rk}\, H_T^\bullet(X^T) = \mathrm{rk}\, H_T^\bullet(X) \le \dim_{\mathbb{Q}} H^\bullet(X) = n$

$\Rightarrow H^\bullet_{T^n}(\mathbb{C}P^{n-1}) = \mathbb{Z}[x, \lambda_1, \dots, \lambda_n] / (x - \lambda_1) \dots (x - \lambda_n)$

<u>Corollary</u>. $\xi \downarrow V^n \rightsquigarrow \begin{array}{c} E \\ \downarrow \mathrm{proj}(V) \\ B \end{array}$ $\quad x = c_1(L^{-1})$ fiberwise over E

<u>Then</u> $\quad B$

$H^\bullet(E) = H^\bullet(B)[x] / x^n - c_1(\xi)x^{n-1} + \dots \pm c_n(\xi)$

Pf: Recall Dold's Lemma - $1, x, \dots, x^{n-1}$ form a basis of $H^\bullet(E)$ as a free $H^\bullet(B)$-module

4. $X = \{0 \subset V^1 \subset V^2 \subset \dots \subset V^{n-1} \subset \mathbb{C}^n\} = U_n / T^n$

$X_{U_n} = BT^n \Rightarrow H^\bullet_{U_n}(X) = \mathbb{Z}[x_1, \dots, x_n]$

$\downarrow U_n / T^n$ $\quad$ which is a free $\mathrm{rk} = n!$ module

BU_n $\quad$ over $H^\bullet_{U_n}(pt) = \mathbb{Z}[c_1, \dots, c_n]$

via $c_k = \sigma_k(x_1, \dots, x_n)$

$\bigoplus V^i / V^{i-1} \cong \mathbb{C}^n \Rightarrow (1 + x_1) \dots (1 + x_n) = 1 + c_1 + \dots + c_n$

$x_i := c_1(V^i / V^{i-1}) \Rightarrow H^\bullet(U^n / T^n) = \mathbb{Z}[x] / (\sigma_1(x), \dots, \sigma_n(x))$

Lecture 26 | Bott's residue formula

G-equivariant push-forward along

$$\begin{array}{ccccc} X & \text{– Closed (oriented) } G\text{-mfd} & X_G & & H^\bullet_G(X) \\ \pi\downarrow & \uparrow \text{connected} & \pi^G \downarrow & & \downarrow \pi^G_! = ? \\ \bullet\ pt & & BG & & H^\bullet(BG) \end{array}$$

Take $T^n \subset G$ - max. torus, $BT = \lim (\mathbb{C}P^N)^n$

$$\begin{array}{ccccc} X_G & \leftarrow & X_T & \supset & X_T^{(N)} \\ X\downarrow & & X\downarrow & & X\downarrow \\ BG & \leftarrow & BT & \supset & (\mathbb{C}P^N)^n \end{array} \qquad \begin{array}{ccccc} \alpha \in H^\bullet(X_G) & \to & H^\bullet(X_T) & \to & H^\bullet(X_T^{(N)}) \\ \uparrow \deg \le 2N & & & & \pi_! \downarrow \\ H^\bullet(BG) & \subset & H^\bullet(BT) & \to & H^\bullet((\mathbb{C}P^N)^n) \\ \pi_!(\alpha) \in & & \uparrow \text{iso in } \deg \le 2N & & \end{array}$$

If $\alpha \in H^\bullet(X_T)$ comes from $H^\bullet(X_G)$, then it is Weyl-group invariant, and so is $\pi_!(\alpha) \in H^\bullet(BT)$

Geometric meaning

$$\alpha \in H^\bullet_T(X) \Rightarrow \pi_!(\alpha) = \sum c_{k_1 \dots k_n} \lambda_1^{k_1} \dots \lambda_n^{k_n}$$

$$\deg \alpha - \dim X = 2(k_1 + \dots + k_n)$$

$$\begin{array}{ccc} L_1 \times \dots \times L_n & \sim \text{associated bundle with } T\text{-space } X \text{ as the fiber} & E \text{ closed oriented mfd} \\ \downarrow & & \pi \downarrow X \\ \mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n} & & \mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n} \end{array}$$

α defines a cohomology class $\alpha_{k_1 \dots k_n} \in H^{\deg \alpha}(E)$

$c_{k_1 \dots k_n} = \langle [\mathbb{C}P^{k_1} \times \dots \times \mathbb{C}P^{k_n}], \pi_!(\alpha_{k_1 \dots k_n}) \rangle$ - Integer representing push-forward of $\alpha_{k_1 \dots k_n}$ from E to pt.

Bott's Fixed Point Localization Formula

$$\boxed{\int_X \alpha = \int_{X^T} \frac{i^* \alpha}{\mathrm{Euler}_T(\nu_i)}}$$

T-equivariant Euler class

X ← T-manifold; $X \supset_i X^T$, i ← inclusion; X^T ← fixed pt submfd; ν_i ← normal bundle to X^T in X

$\int_X$, $\int_{X^T}$ - T-equivariant push-forwards from X (resp. X^T) to pt.

Commentaries

1° X^T is a submfd $\Leftarrow$ Compact group action is locally equivalent to its linearization

Pf: Averageate over G any Riem. metric and use $\underline{\exp}: T_{\circledast} X \to X$ ($\circledast$ fixed pt.)

2° $\mathrm{Euler}_T(\nu_i)$ is invertible over $\mathbb{Q}(\lambda)$

Over a connected component of X_T,

$\nu = \bigoplus_\alpha W_\alpha$, $t \in T$ acts on W_α by character $e^{-2\pi i \chi_\alpha(t)}$, $\chi_\alpha = \ell_1(\alpha)\lambda_1 + \dots + \ell(\alpha)\lambda_n$

If $(1+x_1)\dots(1+x_k) = 1 + c_1(W) + \dots + c_k(W)$ is the (topological) total Chern class of W,

then $\mathrm{Euler}_T(W) = (\chi(\lambda)+x_1)\dots(\chi(\lambda)+x_k)$

$= \chi(\lambda)^k + c_1(W)\chi(\lambda)^{k-1} + \dots + c_k(W)$

non-zero characters since ν is $\underline{\text{normal}}$ to X^T.

$\underline{\text{Proof of Bott' formula}}$: Over $\mathcal{R} = \mathbb{Q}(\lambda)$

Find $\beta \in H^\bullet_T(X^T; \mathcal{R})$ s.t. $i_!(\beta) = \alpha \in H^\bullet_T(X)$.

Then $\pi_! i_!(\beta) = \pi_!(\alpha)$.

But $H^\bullet_T(X; \mathcal{R}) \xrightarrow[i^*]{} H^\bullet_T(X^T; \mathcal{R})$ – $\underline{\text{iso}}$

and $i^*\alpha = i^* i_!(\beta) = \mathrm{Euler}_T(\nu_i) \cdot \beta$

$\Rightarrow \beta = i^*\alpha / \mathrm{Euler}_T(\nu_i)$ Q.E.D.

$\underline{\text{Corollary}}$: $\displaystyle\int_{X^T} \frac{i^*\alpha}{\mathrm{Euler}_T(\nu_i)} \in \mathbb{Z}[\lambda_1, \dots, \lambda_n]$ no denominators

$\underline{\text{Example}}$: $\displaystyle\int_{\mathrm{proj}(\mathbb{C}^n) = \mathbb{C}P^{n-1}} \Phi(x,\lambda) = \operatorname*{Res}_{x=\infty} \frac{\Phi(x,\lambda)\,dx}{(x-\lambda_1)\dots(x-\lambda_n)}$

$$\operatorname*{Res}_{x=\lambda_k} = \frac{\Phi(\lambda_k, \lambda)}{\prod_{j\neq k}(\lambda_k - \lambda_j)} \qquad \Phi(\lambda_k,\lambda) = i^*_{k\text{-th fixed pt}}\Phi, \quad \prod_{j\neq k}(\lambda_k-\lambda_j) = \mathrm{Euler}_T\left(T_{k\text{-th f. p.}} \mathbb{C}P^{n-1}\right)$$

$$T_{k\text{-th f. p.}} \mathbb{C}P^{n-1} = \mathrm{Hom}\Big(\mathbb{C}e_k, \bigoplus_{j\neq k} \mathbb{C}e_j\Big)$$ characters $\chi_j = \lambda_k - \lambda_j$