and deduce (by induction on p), the expansion

$$(2) \qquad S_p(X) = \sum_{n\geqslant 0} \binom{p+n-1}{n} X^n,$$

where $\binom{k}{h}$ denotes the binomial coefficient $\frac{k!}{h!(k-h)!}$.

b) Use $S_p(X) . S_q(X) = S_{p+q}(X)$ to show that

$$(3) \qquad \sum_{0\leqslant l\leqslant n} \binom{p+l+1}{l}\binom{q+n+l-1}{n-l} = \binom{p+q+n+1}{n}$$

(which is a generalisation of (1), the case when $q = 1$).

3. Find the precise form of the polynomials P_n in the proof of proposition 7. 1, § 1, for $n \leqslant 5$ and calculate the terms of degree $\leqslant 5$ of the formal (compositional) inverse series of

$$S(X) = X - \frac{1}{3}X^3 + \frac{1}{5}X^5 + \cdots + (-1)^p \frac{1}{2p+1} X^{2p+1} + \cdots.$$

4. Find the radii of convergence of the following series :

a) $\sum_{n\geqslant 0} q^{n^2} z^n \qquad (|q| < 1),$

b) $\sum_{n\geqslant 0} n^p z^n \qquad (p \text{ integer} > 0),$

c) $\sum_{n\geqslant 0} a_n z^n$, with $a_{2n+1} = a^{2n+1}$, $a_{2n} = b^{2n}$ for $n \geqslant 0$,

where a and b are real and $0 < a, b < 1$.

5. Given two formal power series

$$S(X) = \sum_{n\geqslant 0} a_n X^n \quad \text{and} \quad T(X) = \sum_{n\geqslant 0} b_n X^n \quad (b_n \neq 0),$$

let

$$U(X) = \sum_{n\geqslant 0} (a_n)^p X^n, \quad V(X) = \sum_{n\geqslant 0} a_n b_n X^n, \quad W(X) = \sum_{n\geqslant 0} (a_n/b_n) X$$

(where p is an integer). Prove the following relations :

$$\rho(U) = (\rho(S))^p, \qquad \rho(V) \geqslant \rho(S) . \rho(T),$$

and, if $\rho(T) \neq 0$,

$$\rho(W) \leqslant \rho(S)/\rho(T).$$

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6. Let a, b and c be elements of $\mathbf{C}$, c not an integer $\leqslant 0$. What is the radius of convergence of the series

$$S(X) = 1 + \frac{ab}{c}X + \frac{a(a+1).(b+1)}{2!c(c+1)}X^2 + \cdots + \frac{a(a+1)\ldots(a+n-1).b(b+1)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)}X^n + \cdots$$

Show that its sum $S(z)$, for $|z| < \rho(S)$, satisfies the differential equation

$$z(1-z)S'' + (c - (a+b+1)z)S' - abS = 0.$$

7. Let $S(X) = \sum_{n \geqslant 0} a_n X^n$ be a formal power series such that $\rho(S) = 1$. Put

$$s_n = a_0 + \cdots + a_n, \quad t_n = \frac{1}{n+1}(s_0 + s_1 + \cdots + s_n) \quad \text{for} \quad n \geqslant 0,$$

and put

$$U(X) = \sum_{n \geqslant 0} s_n X^n, \qquad V(X) = \sum_{n \geqslant 0} t_n X^n.$$

Show that : (i) $\rho(U) = \rho(V) = 1$, (ii) for all $|z| < 1$,

$$\frac{1}{1-z}\left(\sum_{n \geqslant 0} a_n z^n\right) = \sum_{n \geqslant 0} s_n z^n.$$

8. Let $S(X) = \sum_{n \geqslant 0} a_n X^n$ be a formal power series whose coefficients are defined by the following recurrence relations :

$$a_0 = 0, \; a_1 = 1, \; a_n = \alpha a_{n-1} + \beta a_{n-2} \quad \text{for} \quad n \geqslant 2,$$

where α, β are given real numbers.

a) Show that, for $n \geqslant 1$, we have $|a_n| \leqslant (2c)^{n-1}$ where $c = \max(|\alpha|, |\beta|, 1/2)$ and deduce that the radius of convergence $\rho(S) \neq 0$.

b) Show that

$$(1 - \alpha z - \beta z^2)S(z) = z, \quad \text{for} \quad |z| < \rho(S),$$

and deduce that, for $|z| < \rho(S)$,

$$S(z) = \frac{z}{1 - \alpha z - \beta z^2}. \qquad (1)$$

c) Let z_1, z_2 be the two roots of $\beta X^2 + \alpha X - 1 = 0$. By decomposing

$$f'(x) = \frac{1}{(x-x_0)^{k+1}}[(x-x_0)g'(x) - kg(x)] = \frac{1}{(x-x_0)^{k+1}}g_1(x),$$

and as $g_1(x_0) \neq 0$, x_0 is a pole of f' of order $k+1$.

Exercises

1. Let K be a commutative field, X an indeterminate and $E = K[[X]]$ the algebra of formal power series with coefficients in K. For S, T in E, define

$$d(S, T) = \begin{cases} 0 & \text{if } S = T, \\ e^{-k} & \text{if } S \neq T, \text{ and } \omega(S - T) = k. \end{cases}$$

a) Show that d defines a distance function in the set E.

b) Show that the mappings $(S, T) \to S + T$ and $(S, T) \to ST$ of $E \times E$ into E are continuous with respect to the metric topology defined by d.

c) Show that the algebra K[X] of polynomials is everywhere dense in E when considered as a subset of E.

d) Show that the metric space E is complete. (If (S_n) is a Cauchy sequence in E, note that for any integer $m > 0$, the first m terms of S_n do not depend on n for sufficiently large n.)

e) Is the mapping $S \to S'$ (the derivative of S) continuous?

2. Let p, q be integers $\geqslant 1$. Let $S_1(X)$ be the formal series

$$1 + X + X^2 + \cdots + X^n + \cdots,$$

and put

$$S_p(X) = (S_1(X))^p.$$

a) Show, by induction on n, that

$$(1) \quad 1 + p + \frac{p(p+1)}{2!} + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!} = \frac{(p+1)\ldots(p+n)}{n!},$$

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