Math 53. Multi-variable Calculus. Problems from past final exams

1. Find and classify all critical points of the function $f(x,y) = y^3 + 3x^2y - 12y$ as local maxima, local minima or saddle points.

2. Find the maximum and minimum values of the function $x + y + z$ on the surface of the ellipsoid

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{6} = 12.$$

3. Change the order of integration in the triple integral:

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\ dz\ dy\ dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\ dy\ dx\ dz\ .$$

4. A homogeneous ball of radius R and mass density μ rotates with the angular velocity ω about a line passing through its center. Compute the kinetic energy accumulated in the ball due to this rotation.

5. An ink spot of initially round shape $(x-1)^2 + y^2 \leq a^2$ is carried by the flow of a two-dimensional fluid with the velocity vector field

$$\vec{V} = (x + e^{-x^2}\cos y)\vec{i} + (2y + 2xe^{-x^2}\sin y)\vec{j}.$$

Find the area of the ink spot at the moment t.

6. Calculate the area of the surface

$$z = a\left(1 - \frac{2x^2}{a^2} - \frac{2y^2}{a^2}\right), \quad z \geq 0.$$

7. Compute the flux of the vector field $\vec{F} = \vec{r}/|\vec{r}|^3$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, across the surface $(x+1)^2 = y^2 + z^2,\ x \leq 0$, equipped with an orientation of your choice.

8. Give an example of a divergence-free vector field which is not the curl of any vector field. Justify your answer.

9. Find the domain D on the plane for which the line integral

$$\int_{\partial D} (x^2 y - 2y)dx + (2x - y^2 x)dy$$

takes on its maximal possible value and compute this value.

10. On a grid paper, two ants crawl with velocities $(2, 5)$ and $(-2, 3)$, starting at the initial positions $(0, 0)$ and $(15, 0)$ respectively. Find the shortest distance between the ants.

11. A comet follows an elliptical orbit with the major semiaxis 13 $a.u.$ and the minor semiaxis 5 $a.u.$ Find how close the comet comes to the Sun.

("a.u." means "astronomical unit," the average distance from the Earth to the Sun, approximately equal to $150,000,000$ km; this problem assumes familiarity with Kepler's laws of planetary motion, in particular with his 2nd law according to which the Sun is located at one of the foci of the elliptic orbit.)

12. Find out if the cross section of the quadratic surface $2x^2 = 3y^2 + 5z^2 - 1$ by the plane $z = x - y$ is a hyperbola.

13. For a square-shaped thin board of mass density σ and area a^2, compute the moment of inertia about the axis perpendicular to the plane of the square and passing through one of its vertices.

14. Compute the area of the part inside the cylinder $x^2 + y^2 = 12a^2$ of the surface obtained by rotating the parabola $z = x^2/a$ lying in the plane $y = 0$ about the z-axis.

15. Give an example of a curl-free vector field which is not the gradient of any function.

16. A smooth vector field $\vec{F}$ has zero flux across any closed surface. Prove that $\operatorname{div} \vec{F} = 0$.