Math H53. Final Exam. Monday, May 13, 2013

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1. Consider function $f = x^2 - 3xy + 3y^2$ in the region $D : x^2 - xy + y^2 \le 3$.

(a) Prove that the boundary curve ∂D is an ellipse, and that the region D is bounded.

Quadratic form $ax^2 + 2bxy + cy^2 = x^2 - xy + y^2$ is positive (since the determinant $ac = 1 \times 1 > b^2 = (1/2)^2$, and $a > 0$). Therefore positive level curves are ellipses, and D is the interior of such an ellipse.

(b) Find principal axes of the ellipse ∂D and compute the area of D.

The quadratic form has two simmetry lines: $y = x$ and $y = -x$. The boundary ∂D meets these lines at the points where $x^2 - x^2 + x^2 = 3$ and $x^2 + x^2 + x^2 = 3$ respectively, that is $y = x = \pm\sqrt{3}$ and $y = -x = \pm 1$. The semiaxes have lengths $\sqrt{6}$ and $\sqrt{2}$ respectively. Thus the ellipse is obtained from the unit circle by stretching $A = \sqrt{6}$ and $B = \sqrt{2}$ times in two perpendicular directions, and thus encloses the area $\pi AB = 2\pi\sqrt{3}$.

(c) Find critical points of f restricted to the boundary ∂D.

Put $F(x, y, \lambda) := x^2 - 3xy + 3y^2 - \lambda(x^2 - xy + y^2 - 3)$. Under the constraint $x^2 - xy + y^2 = 3$, we have:

$$\begin{cases} 2x - 3y & = \lambda(2x - y) \\ -3x + 6y & = \lambda(-x + 2y) \end{cases} \iff \begin{cases} 2(1-\lambda)x & = (3-\lambda)y \\ (\lambda - 3)x & = 2(\lambda - 3)y \end{cases}$$

If $\lambda = 3$, then $x = 0$, and $y = \pm\sqrt{3}$. If $\lambda \ne 3$, then $x = 2y$ (from the 2nd equation), and from the equation of ∂D, we find $y = \pm 1$ (while from the 1st equation we find: $4 - 4\lambda = 3 - \lambda$, i.e. $\lambda = 1/3$). Thus there are four critical points of $f|_{\partial D}$: $(x, y) = (0, \pm\sqrt{3})$ and $(x, y) = \pm(2, 1)$.

(d) Locate critical points of f inside D and determine if they are local minima, maxima, or saddles.

Quadratic form $ax^2 + 2bxy + cy^2 = x^2 - 3xy + 3y^2$ has determinant $ac - b^2 = 1 \times 3 - (3/2)^2 > 0$ and $a > 0$. Thus it is positive, with the critical point at the origin, which is a (local and global) minimum.

(e) Find the maximum and minimum values of f in D.

The ctirical values of $(x^2 - 3xy + 3y^2)|\partial D$ are: 9 at the critical points $(x, y) = (0, \pm\sqrt{3})$, and $4 - 6 + 3 = 1$ at the critical points $(x, y) = \pm(2, 1)$. The value at the interior critical point is 0. Thus, the minimum value of f in D is 0 (and is acheved at the origin), and the maximum value is 9.

(f) Orient ∂D as the boundary of D, and compute the circulation of ∇f along ∂D and the flux of ∇f across ∂D.

$\nabla f = (2x - 3y)\vec{i} + (-3x + 6y)\vec{j}$. This vector field is conservative, and hence has zero circulation along the closed path ∂D. Its divergence is constant and equal $2 + 6 = 8$. Thus the flux of ∇f across ∂D is equal to 8 times the area of D, i.e. $16\pi\sqrt{3}$.

(g) Denote by $D(t)$ the region into which D is carried by the flow of the vector field ∇f after time t. Compute the area $A(t)$ of $D(t)$.

We have: $dA(t)/dt = 8A(t)$. Thus $A(t) = e^{8t}A(0)$ where $A(0) = 16\pi\sqrt{3}$.

2. The following questions are about differentiable vector fields in three dimensions.
(a) Give an example of a vector field which is curl-free (i.e. has zero curl) but is not conservative. (Don't forget to prove that it is not conservative.)

$$\vec{F} = \frac{-y}{x^2 + y^2}\vec{i} + \frac{x}{x^2 + y^2}\vec{j} + 0\vec{k}$$

is curl-free since it is locally the gradient of cylindrical $\theta = \tan^{-1}(y/x)$. The work of $\vec{F}$ along a circle going arond the z-axis is equal to 2π. Therefore the vector field is not conservative globally.

(b) Give an example of a vector field whose flux across the boundary of every region is equal to the volume of that region. Justify your answer.

$\vec{V} = x\vec{i}$ has the requisite property since by Gauss' theorem

$$\int\int_{\partial R} \vec{V} \cdot d\vec{S} = \int\int\int_R \operatorname{div} \vec{V} dxdydz = \int\int\int_R 1dxdydz = Volume(R).$$

(c) Give an example of a non-zero vector field which is curl-free and divergence-free (i.e. has zero curl and zero divergence). In the domain of this vector field:

- Does there exist a closed oriented curve C such that the circulation of the vector field along C is non-zero? Why?
- Does there exist a closed oriented surface S such that the flux of the vector field across this surface is non-zero? Why?

Take a non-zero constant vector field, ω. It has zero curcl and zero diverhence. It is the gradient of the linear function $\omega \cdot \vec{r}$, and hence its circulation along any closed curve is zero. Its flux across any closed surface is zero. One can justify it using the divergence theorem, applied to the region swept by rays from the origin to points of S. Alternatively, one can use Stokes' Theorem using, representing ω as the curl of another vector field, as it is done in the next question.

(d) Let $\vec{\omega}$ be a constant vector field. Give an example of a divergence-free vector field whose curl is $\vec{\omega}$.

- Does there exist a closed oriented curve C such that the circulation of the vector field along C is non-zero? Why?
- Does there exist a closed oriented surface S such that the flux of the vector field across this surface is non-zero? Why?

Take $\vec{V} = \omega \times \vec{r}/2$. Then (as we've checked in class) $\operatorname{curl} \vec{V} = \omega$. When $\omega \neq \vec{0}$, the circulation of $\vec{V}$ along a radius-R circle perpendicular to ω will be equal to the flux of ω through the disc bounded by this circle, i.e. to $\pm\pi R^2|\omega| \neq 0$. Also, this vectors field has zero divergence (because its flow describes rotations of space about the axis ω passing through the origin, and rotations are volume-preserving). The flux of this vector field over a closed surface is zero (as it fiollows from Gauss' theorem applied to the region formed by rays from the origin with the endpoints on the surface).

3. This problem is about areas of parametric surfaces in space.

(a) Give the definition of surface area of a parametric surface. (*Remark:* The answer should have the form of a double integral.)

Let $(u, v) \mapsto \vec{r}(u, v)$ be a parameterization of surface S is space by a domain D in the plane. Then

$$Area(S) := \int\int_D |\vec{r}_u \times \vec{r}_v| \, dudv.$$

(b) Derive an integral formula for the area of the surface of revolution described in cylindrical coordinates by equation $r = f(z)$, where f is a given differentiable function on the interval $a \le z \le b$. (*Remark:* The answer should have the form of a single-variable integral.)

Parameterize the surface by $(u, v) = (\theta, z)$:

$$x = f(v) \cos u, \;\; y = f(v) \sin u, \;\; z = v.$$

Then

$$\vec{r}_u x \vec{r}_v = (-f \sin u, f \cos u, 0) \times (f' \cos u, f' \sin u, 1) = (-ff', f \cos v, f \sin v).$$

Thus

$$Area(S) = \int_a^b \int_0^{2\pi} \sqrt{1 + (f'(v))^2} f(v) dudv = 2\pi \int_a^b \sqrt{1 + (f')^2} f dv.$$

(c) Compute the area of the surface described by the equation $z^6 = x^2 + y^2$ and inequalities $0 \le z \le 1/2$. (*Remark:* The answer happens to have the form $\pi a / 3^3 4^3$, where a is a 2-digit integer. State your answer by finding a.)

The surface has cylindrical equation $r = z^3$, and by part (b), $Area(S) =$

$$2\pi \int_0^{1/2} \sqrt{1 + 9z^4} z^3 dz = \frac{2\pi}{9 \cdot 4} \int_0^{1/2} \sqrt{1 + 9z^4} d(1 + 9z^4) = \frac{\pi}{3^3} (1 + 9z^4)^{3/2} |_0^{1/2} = \frac{5^3 \pi}{3^3 4^3} - \frac{\pi}{3^3}$$

Thus, $a = 5^3 - 4^3 = 125 - 64 = 61.$