Practice and sample problems from past exams

1. Which of the following expressions make sense?

$$((\vec{a}\times\vec{b})\cdot\vec{c})\times\vec{d},\quad (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}),\quad (\vec{a}\times(\vec{b}\cdot\vec{c}))\times\vec{d},\quad \vec{a}\times((\vec{b}\cdot\vec{c})\times\vec{d}),\quad \vec{a}\times(\vec{b}\cdot(\vec{c}\times\vec{d}))$$

2. Which of the following expressions are equal to zero?

$$(\vec{i}+\vec{k})\cdot(\vec{k}-\vec{i}),\quad (\vec{i}\times\vec{j})\times\vec{k},\quad (\vec{i}\times\vec{k})\times\vec{i},\quad (\vec{i}\times\vec{j})\cdot(\vec{i}+\vec{j}),\quad \vec{k}\times\vec{j}-\vec{j}\times\vec{k}$$

3. Show that if $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, then $\vec{b}=\vec{c}$, provided that $\vec{a}\neq\vec{0}$.

4. Find the angle between two disjoint diagonals in two adjacent faces of a cube.

5. Which of the functions can be transformed into each other by linear changes of coordinates: $3xy+y^2$, $x^2+3xy+2y^2$, $x^2+3xy-2y^2$?

6. Describe all those quadratic curves which *cannot* be obtained as conic sections.

7. A scalar-valued function f of a vector argument is called *linear* if

$$f(\lambda\vec{u}+\mu\vec{v})=\lambda f(\vec{u})+\mu f(\vec{v})$$

for all vectors $\vec{u},\vec{v}$ and all scalars λ,μ.

Let $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$. Which of the following functions of $\vec{r}$ are linear:
(a) $f(\vec{r})=2+3x+4y+5z$, (b) $f(\vec{r})=|\vec{r}|$, (c) $f(\vec{r})=\vec{r}\cdot\vec{r}$, (d) $f(\vec{r})=\vec{a}\cdot(\vec{r}\times\vec{b})$, where $\vec{a},\vec{b}$ are given vectors?

8. Express the dot-product $\vec{a}\cdot\vec{b}$ of two vectors through the lengths M and N of their sum and difference: $M=|\vec{a}+\vec{b}|$ and $N=|\vec{a}-\vec{b}|$.

9. Compute the area of the quadrilateral $ABCD$ with the vertices:

$$A=(0,0,0),\ B=(1,2,3,),\ C=(3,5,8),\ D=(2,3,5).$$

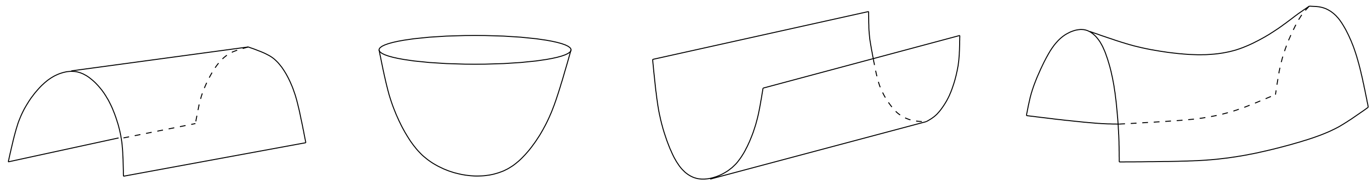
10. A function f on the plane is said to have a given line as an *axis of symmetry* if $f(P)=f(Q)$ whenever the points P and Q are symmetric to each other about this line. Find all axes of symmetry of the function $f(x,y)=x^2+y^2-4xy$.

11. How many axes of symmetry does the function x^2+y^2 have?

12. Any ellipse can be described as the set of all points in the plane with a fixed sum of distances to two fixed point, called *foci*. Locate the foci of the ellipse $ax^2+by^2=1$ assuming that $a>b$.

13. A comet follows an elliptical orbit with the major semiaxis 13 *a.u.* and the minor semiaxis 5 *a.u.* Find how close the comet comes to the Sun ("a.u." means "astronomical unit," the average distance from the Earth to the Sun, approximately equal to $150,000,000$ *km*.)

14. Identify graphs of the functions: $A = x^2+y^2-2xy$, $B = x^2+y^2-xy$, $C = x^2+y^2$, $D = x^2+y^2+2xy$, $E = x^2+y^2+3xy$.



15. Can the following pairs of surfaces occur as level sets of the same quadratic form in three variables (if "yes" give an example, if "no", explain why):
(a) a sphere, and an ellipsoid which is not a sphere?
(b) a hyperboloid of one sheet and a hyperboloid of two sheets?
(c) a cylinder and the empty set? (d) a cone and a cylinder?
(e) a cone and a hyperboloid of one sheet?

16. On a grid paper, two ants are crawling with velocities $(2,5)$ and $(-2,3)$, starting at the initial positions $(0,0)$ and $(15,0)$ respectively. Find the shortest distance between the ants.

17. Astronomers observe a strange planet, Swingus, moving around the Sun $(0,0,0)$ according to the law: $x = a\cos t,\ y = b\sin t,\ z = 0$. Describe the orbit of Swingus, and find out if the motion obeys the angular momentum conservation law.

18. Fill-in the blanks using formulas A, B, C, D, E, F, G. Multiple use of the same formula is allowed.

`Theorem:` *The arc length of a parametric curve does not change under reparameterization.*

`Proof.` Given a parametric curve ..., its arc length is defined by .... Given a monotone reparameterization ..., the reparameterized curve is: .... By the chain rule, the velocity of the reparameterized curve is given by the formula: .... The arc length of the reparameterized curve is therefore defined as .... Since the reparameterization is monotone, ... does not change its sign. Thus, applying to the former integral ... the change of variables ..., we obtain the latter integral ....

A. $dt/d\tau$ **B.** $t = t(\tau)$ **C.** $\int_a^b |\dot{\vec{r}}|(t)dt$ **D.** $\dot{\vec{r}}(t(\tau))dt/d\tau$ **E.** $\int_\alpha^\beta |\dot{\vec{r}}||dt/d\tau|d\tau$
F. $[a,b] \to \mathbb{R}^3 : t \mapsto \vec{r}(t)$ **G.** $[\alpha,\beta] \to \mathbb{R}^3 : \tau \mapsto \vec{r}(t(\tau))$

19. Show that the length of the graph of a differentible function $f : [a,b] \to \mathbb{R}$ is given by $\int_a^b \sqrt{1+(f'(x))^2}dx$.