THREE POINTS ON THE PLANE

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In this lecture, we examine some properties of a geometric figure formed by three points on the plane.

Any three points not lying on the same line determine a *triangle*. Namely, the points are the *vertices* of the triangle, and the segments connecting them are the *sides* of it.

We expect the reader to be familiar with a few basic notions and facts of geometry of triangles. References like "[K], §140" will point to specific sections in:

Kiselev's Geometry. Book I: Planimetry, Sumizdat, 2006, 248 pages.

Everything we assume known, as well as the initial part of the present exposition, is contained in the first half of this textbook.

Circumcenter. One says a circle is *circumscribed about* a triangle (or shorter, is the *circumcircle* of it), if the circle passes through all vertices of the triangle.

`Theorem 1.` ***About every triangle, a circle can be circumscribed, and such a circle is unique.***

Let $\triangle ABC$ be a triangle with vertices A, B, C (Figure 1). A point O equidistant from[1] all the three vertices is the center of a circumcircle of radius $OA = OB = OC$. Thus we need to show that a point equidistant from the vertices exists and is unique.

Indeed, the *geometric locus* [2] of points equidistant from A and B is the *perpendicular bisector* to the segment AB, i.e. it is the line (denoted by MN in Figure 1) perpendicular to the segment AB at its midpoint (see [K], §56). Likewise, the geometric locus of points equidistant from B and C is the perpendicular bisector PQ to the segment BC. Let O denote the point of intersection of the lines MN and PQ. Such a point exists, since if MN and PQ were parallel, the segments AB and BC perpendicular to them would lie in the same line (see [K], §80), which case is assumed to be excluded. Of course, the intersection point O of MN and PQ unique. Then O is equidistant from A and B, and to B

[1] "Equidistant from" is a shorthand for "is the same distance away from".

[2] It is formed by all those points which possess a property in question.

and C, i.e. it is equidistant from all the vertices of $\triangle ABC$. Thus the circle of radius OA centered at O is the unique circumcircle of $\triangle ABC$.

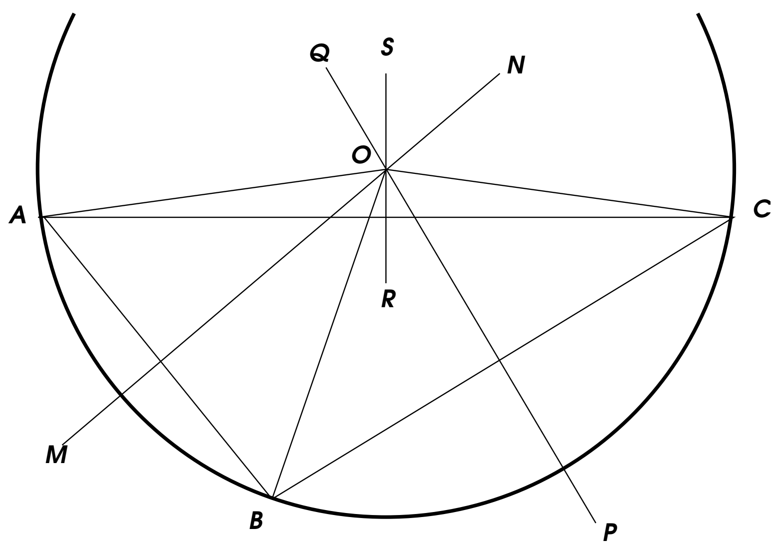


Figure 1

Notice that the center O of the circumcircle, being equidistant from C and A, lies on the perpendicular bisector RS to the segment CA. We conclude that all the three perpendicular bisectors intersect at O.

Three or more lines intersecting at a point are said to be *concurrent*. We have obtained the following corollary.

Corollary. ***Perpendicular bisectors to the sides of a triangle are concurrent.***

Moreover, the intersection point of the three perpendicular bisectors to sides of a triangle is the center of its circumcircle, also called the *circumcenter* of the triangle.

Orthocenter. The perpendicular, dropped from a vertex of a triangle, to the side (or an extension of it) opposite to the vertex is called an *altitude* [3] of the triangle.

Theorem 2. ***In every triange, the three altitudes are concurrent.***

[3] or a *height* of the triangle

Indeed, through each vertex of $\triangle ABC$ (Figure 2), draw the line parallel to the opposite side of the triangle. These lines form an auxiliary triangle $A'B'C'$ such that the midpoints of its sides are vertices of $\triangle ABC$. Indeed, $C'A = BC$ and $AB' = BC$ as opposite sides of the parallelograms $C'BCA$ and $ABCB'$ respectively (see [K], §85). Therefore $C'A = AB'$, i.e. A is the midpoint of $B'C'$, and similarly B and C are midpoints of the other two sides of $\triangle A'B'C'$. The altitudes AD, BE, and CF of $\triangle ABC$ turn out to be perpendicular bisectors to the sides of $\triangle A'B'C'$ and are therefore concurrent by Corollary of Theorem 1.

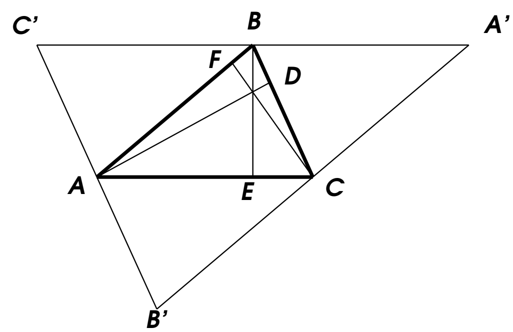


Figure 2

The intersection point of the altitudes of a triangle is called its *orthocenter.*

`Exercises.` (a) Prove that the circumcenter lies inside the triangle if and only if the triangle is *acute* (i.e. has three acute angles), and outside the triangle if and only if the triangle is *obtuse* (i.e. has one obtuse angle).

(b) Find out which triangles have their orthocenter lie inside (respectively outside) the triangle.

(c) Can the circumcenter (respectively orthocenter) lie on one of the sides of the triangle?

Barycenter. The segment connecting a vertex of a triangle with the midpoint of the opposite side is called a *median* of the triangle.

`Theorem 3.` ***In every triangle, the three medians are concurrent.***

Indeed, in a triangle ABC (Figure 3), denote M the intersection point of the medians AD and BE. Let F and G be the midpoints of AM and BM respectively. Then DE is a midline of $\triangle ABC$, and FG is a midline of $\triangle ABM$. By the property of the midline (see [K],

§95), each of the segments DE and FG are parallel to the segment AB and congruent to a half of it. Therefore DE and FG are parallel and congruent to each other, and hence $DEFG$ is a parallelogram (see [K], §86). Thus M, being the intersection point of the diagonals of the parallelogram $DEFG$, is its center of symmetry (see [K], §89). In particular, M is the midpoint of the diagonal FD, i.e. $DM = MF = FA$. We conclude that the point M at which the median BE meets the median AB divides AB in the proportion $2:1$ counting from the vertex A. Since the same will be true for the point at which the median CH meets the median AB, we find that the median CH passes through the same point M.

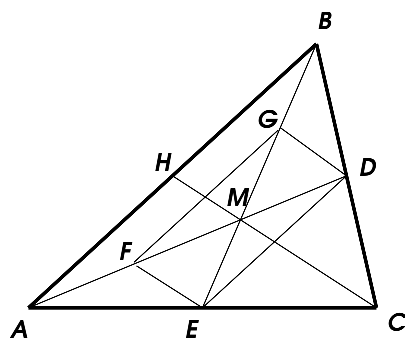


Figure 3

The intersection point of the three medians of a triangle is sometimes called the *barycenter* of the triangle. From the proof of the theorem we obtain the following corollary.

Corollary. ***The barycenter of the triangle divides each median in the proporion*** $2:1$ ***counting from the vertex.***

Exercises. (d) Show that the median CH (Figure 3) of $\triangle ABC$ is parallel to the sides EF and GD of the parallelogram $DEFG$.

(e) Prove that barycenter of the triangle formed by midlines of a given triangle coincides with the barycenter of the given one.

(f) Prove that midpoints of the sides of a given triangle and the midpoints of the segments connecting the vertices of the given triangle with its barycenter form two triangles *centrally symmetric* (see [K], §88) about the barycenter.

The *homothety with the coefficient* $-1/2$ *about a center* O is a geometric transformation that can be described as follows (see [K], §180). A point A' is obtained by this transformation from a point A if the segment AA' contains the point O and is divided by it in the proportion

$2:1$, i.e. if $OA' = OA/2$. Two figures, $\mathcal{F}$ and $\mathcal{F}'$, are said to be homothetic with the coefficient $-1/2$ about the center O, if each point of $\mathcal{F}'$ is obtained by this transformation from a point of $\mathcal{F}$ and moreover, $\mathcal{F}'$ contains all points obtained from points of $\mathcal{F}$ by this transformation. Similarly to central symmetry, the homothety transforms a line into a parallel (or the same) line, and every angle into an angle congruent to it. However, unlike central symmetry (which also transforms each segment to a segment congruent to it) the homothety with the coefficient $-1/2$ transforms each segment into a parallel segment which is twice as shorter than the original one. Using the concept of homothety, we can formulate the previous corollary this way.

`Corollary'`. ***The midlines of a given triangle form the triangle homothetic with the coefficient*** $-1/2$ ***to the original one about its barycenter.***

`Exercises.` (g) Show that $\triangle ABC$ in Figure 2 is obtained from the $\triangle A'B'C'$ by a homothety with the coefficient $-1/2$, and find the center of this homothety.

(h) Describe the geometric transformation *inverse* [4] to the homothety with the coefficient $-1/2$ about a given center. (It is called the homothety with the coefficient -2 about the same center.)

(i) Generalize the concept of homothety with coiefficients $-1/2$ and -2 to the case of arbitrary coefficent k, and show that central symmetry is the homothety with the coefficient -1.

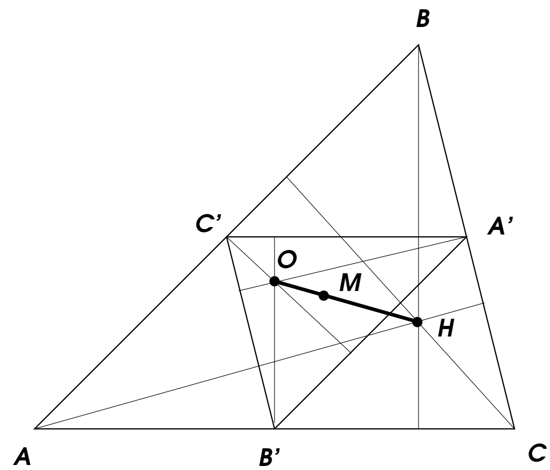
Euler's line. Three or more points lying on the same line are said to be *collinear.*

`Theorem 4`. ***The circumcenter, orthocenter, and barycenter of every triangle are collinear.***

Indeed, to a given triangle ABC (Figure 4), apply the transformation of homothety with the coefficient $-1/2$ about its barycenter M to obtain the triangle $A'B'C'$ formed by the midpoints of the sides. The altitudes of $\triangle ABC$ will transform into the altitudes of $\triangle A'B'C'$ which are perpendicular bisectors to the sides of $\triangle ABC$. Therefore the orthocenter H of $\triangle ABC$ will transform into its circumcenter O. Thus the circumcenter is homothetic with the coefficient $-1/2$ to the orthocenter about the barycenter. In particular, the three points are collinear.

[4] I.e. undoing whatever the direct transformation is doing.

`Corollary.` ***The barycenter*** M ***of the triangle divides the segment*** HO ***connecting the orthocenter with the circumcenter in the proportion*** $2 : 1$***.***



`Figure 4`

`Exercise.` (j) Show that if two of the three centers O, H, M of $\triangle ABC$ coincide, than all three coincide, and derive that in this case the triangle is equilateral.

Unless the triangle is equilateral, the orthocenter, circumcenter and baricenter are distinct, and hence the line containing them is uniquely defined. This line is called *Euler's line* of the triangle.

Euler's circle. By definition, *Euler's circle* of a given triangle is the circumcircle of the triangle formed by its midlines (see Figure 5).

`Exercise.` (k) Show that the radius of Euler's circle is a half the readius of the circumcircle.

`Theorem 5.` ***The center of Euler's circle lies on Euler's line and bisects the segment between the orthocenter and circumcenter. Euler's circle passes through the following*** 9 ***remarkable points of the triangle: the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments connecting the orthocenter with the vertices of the triangle.***

Indeed, according to Corollary' of Theorem 3, the triangle formed by the midlines is homothetic with the coefficent $-1/2$ to the given triangle about the barycenter. Consequently, Euler's circle is obtained by the same transformation from the circumcircle of the given triangle

. In particular, the center E of Euler's circle is homothetic with the coefficient $-1/2$ about the barycenter M to the circumcenter O of the given triangle. It follows that E lies on Euler's line. Moreover, since M divides HO and OE in the proportion $2 : 1$, we find that $EO = MO + ME = HO/3 + HO/6 = HO/2$, i.e. that E bisects HO.

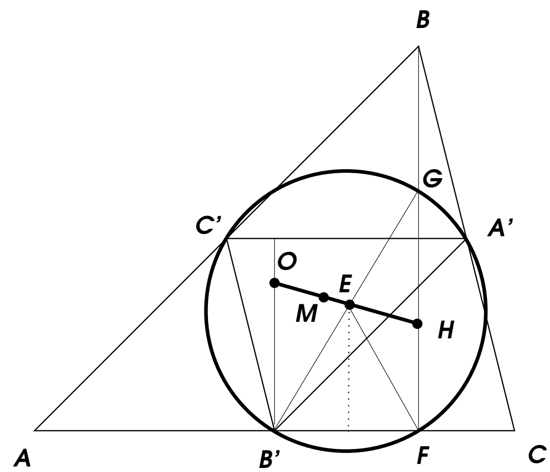


Figure 5

Furthermore, the perpendicular bisector $B'O$ to the side AC, and the altitude BF perpendicular to this side, are parallel to each other. Hence the perpendicular bisector to the segment $B'F$ (it is shown as the dotted line in Figure 5), being the midline the trapezoid $B'OHF$ (see [K], §97), contains the midpoint E of HO. Thus E is equidistant from B' and F.

Let us now extend the segment $B'E$ past the point E and denote by G the intersection point with the altitude BF. In the triangles OEB' and HEG, the angles OEB' and HEG are congruent (as vertical, see [K], §26), the angles $B'OE$ and GHE are congruent (as interior alternate angles formed by parallel lines and a transversal, [K], §77), and the sides OE and HE, to which those angles are adjacent, are congruent too. Thus $\triangle OEB'$ and $\triangle HEG$ are congruent by the ASA-test (see [K], §40). Therefore $EB' = EG$ and $OB' = HG$ as respective sides in congruent triangles. Since OB' is homotheic with the coefficent $-1/2$ to HB, we find that $HG = OB' = HB/2$, i.e. that G is the midpoint of the segment HB.

Finally, since $EG = EB' = EF$, where E is the center of Euler's circle, and EB' is a radius of it, we see that B', F and G lie on Euler's circle. Thus we have proved that the midpoint B' of the side AC, the foot F of the altitude dropped to this side, and the midpoint G of the segment connecting the orthocenter H with the vertex opposite to this side, lie on Euler's circle.

The same conclusion holds true for the corresponding 3 points associated with each of the other two sides of the triangle.

Euler's circle is often called the *nine-point circle* of the triangle.

Incenter. A circle is said to be *inscribed into* a given triangle if it is tangent to each of its sides.

Theorem 6. ***Into every triangle, a circle can be inscribed, and such a circle is unique.***

Let $\triangle ABC$ be a triangle into which a circle with the center K is inscribed (Figure 6). Then K is equidistant from all the three sides of the triangle, i.e. the perpendiculars KP, KQ, and KR dropped from the center to the sides are congruent to each other. Thus we need to show that the point K such that $KP = KQ = KR$ exists and is unique.

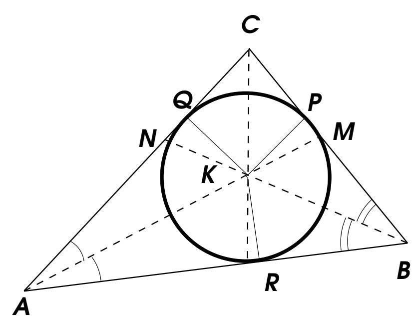


Figure 6

Indeed, the geometric locus of points equidistant from the sides AB and AC is the bisector AM of the angle A (see [K], §56). Likewise, the geometric locus of points equidistant from the sides BA and BC is the bisector BN of the angle B. Let K denote the intersection point of AM and BN. Such a point exists since the angle bisectors AM and BN are not parallel. For if they were, the angles BAM and ABN would have add up to 180 degrees as interior same-side angles formed by parallel lines and a transversal (see [K], §77), which isn't the case. Of course, the intersection point K is unique. It is equidistant from AB and AC, and from BA and BC, i.e. from all the sides of $\triangle ABC$.

Note that the center K of the incircle, being equidistant from CA and CB, lies also on the bisector of the angle C. We arrive at the following corollary.

`Corollary.` ***Angle bisectors of a triangle are concurrent.***

Moreover, the intersection point of the angle bisectors of a triangle is the center of its incircle, also called the *incenter* of the triangle.

Escribed circles. A circle tangent to one side of a triangle and to extensions of the other two sides are called *escribed* circles of the triangle. For every triangle, there are 3 such circles, and they lie outside the triangle (Figure 7). To construct them, draw the bisectors of the exterior angles of a triangle ABC, and find their pairwise intersection points K_A, K_B, and K_C. Each of these point is equidistant from one side of the triangle and extensions of the other two sides (e.g. K_A is equidistant from BC and extensions of AC and AB). Thus each of these points is the center of an escribed circle. The radius of it is the perpendicular dropped from the point to any of the lines AB, BC, and AC.

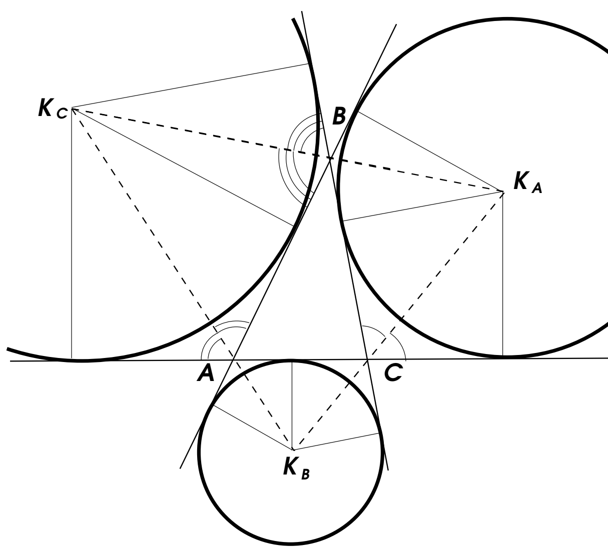


Figure 7

`Exercises.` (l) Show that the bisector of any interior angle of a triangle is concurrent with the bisectors of the exterior angles adjacent to the other two vertices.

(m) Prove that the incenter of a $\triangle ABC$ (Figure 7) is the orthocenter of $\triangle K_A K_B K_C$ formed by the bisectors of its exterior angles.

(n) Prove that the orthocenter of a given triangle is the incenter of another triangle whose vertices are the feet of the altitudes of the given triangle.

Feuerbach's theorem. We conclude the lecture with the formulation of the following amazing fact.

Theorem. ***For every triangle, the nine-point circle is tangent to the inscribed circle and to each of the three escribed circles.***

We don't know any proof of this theorem that would fit the elementary style of our exposition. It would not be too hard (although quite messy) to verify Feuerbach's theorem using coordinates, by computing the distance between the center E of Euler's circle and K_A, K_B, K_C or K, and discovering that it is equal to the sum (respectively difference) of the radii of the circles claimed to be tangent to each other. A more intelligent proof, based on the *method of inversion*, can be found, for instance, at

http://www.cut-the-knot.org/Curriculum/Geometry/FeuerbachProof.shtml

It would be interesting to find a more elementary proof, and even more intersecting to figure out how the fact could have been discovered.