Examples and exercises for Lecture 1.

1. HRR on Riemann surfaces. Let Σ be a compact non-singular complex curve (= Riemann surface) of genus g, and L a holomorphic line bundle over it of degree d (that is, the difference between the number of zeroes and poles of a mromorphic section of it is equal to d). Apply the Hirzebruch-Riemann-Roch formula to express the holomorphic Euler characteristic $\chi(\Sigma, L) := \dim H^0(\Sigma, L) - \dim H^1(\Sigma, L)$ as $d - g + 1$.

2. Lefschetz' fixed point formula on complex projective spaces. Let Λ be an invertible diagonal matrix acting on $\mathbf{C}^n$ with eigenvalues Λ_j, and P the Hopf line bundle over $M = \mathrm{proj}(\mathbf{C}^n)$ with the natural action of Λ. Show that

$$\mathrm{str}\left(\Lambda : H^*(M; P^{\otimes k})\right) = -\operatorname{Res}_{P=0,\infty} \frac{P^k}{(1-\Lambda_1^{-1}P)\cdots(1-\Lambda_n^{-1}P)} \frac{dP}{P}.$$

Hint. Recalculate the sum of residues at $P = 0$ and $P = \infty$ as the sum of residues at $P = \Lambda_j$ and compare with Lefschetz' fixed point contributions.

Derive that the str vanishes when $k = 1, ..., n-1$. Could you explain this result using Kodaira's vanishing theorem?

Exercises to Lecture 2.

1. Rigidity of arithmetical genus. As was mentioned in class, given a holomorphic $\mathbf{C}^\times$-action on a compact complex manifold M, one can define the "equivariant arithmetical genus" of M as the function on the complex circle $\mathbf{C}^\times$: $\Lambda \mapsto \mathrm{str}(\Lambda : H^*(M, \mathcal{O}_M)$, where $\mathcal{O}_M$ is the structure sheaf of M. Using Lefschetz' holomorphic fixerd point formula, prove that the function is constant (and is equal therefore to the arithmetical genus $\chi(M, \mathcal{O}_M)$).

2. (Equivariant) HRR on projective spaces. On $M = \mathrm{proj}(\mathbf{C}^n)$ reconcile two localization formulas (cohohomological and K-theoretic ones) with the (equivariant version) of HRR. Namely, by such HRR,

$$\chi_T(M, P^k) = \int_M e^{-kp}\, \mathrm{td}_T(T_M)$$

where td_T is the torus-equivariant Todd class, and e^{-kp} (p being the torus-equivariant 1st Chern class of the hyperplane line bundle $\mathcal{O}(1)$) is the Chern character of P^k, the k-th power of the Hopf bundle. The R.H.S. can be expressed using Bott's localization formula

$$\int_M F(p) = -\operatorname{Res}_{p=\infty} \frac{F(p)dp}{(p-\lambda_1)\cdots(p-\lambda_n)},$$

while the L.H.S. can be expressed by Lerfschetz' formula as in Exercise 2 of Lecture 1. Identify the residue expressions (and be prepare to fight an extra-factor n that seemigly ruins the whole of mathematical knowledge).

3. A "counter-example". Give an example of a *free* S^1-action on some M such that $H^*_{S^1}(M)$ has positive rank over $H^*_{S^1}(pt)$ in apparent contradiction to the claim that the rank of circle-equivariant cohomology is supported at the fixed point locus.

4. Flag manifolds. For $G = U_n$, and $M = U_n/T^n$ (the manifold of complete flags in $\mathbf{C}^n$), compute $H^*_G(M)$ and $K^*_G(M)$.