LINEAR ALGEBRA. Part I

`Definitions.` A *quadratic form* on the vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a homogeneous degree-2 polynomial $Q(\mathbf{x}) = \sum_{i,j} q_{ij} x_i x_j$, where we may assume that $q_{ij} = q_{ji}$ for all $i, j,$) and where $(x_1, \dots, x_n)$ are coordinates of the vector $\mathbf{x}$. A *bilinear form* on the vector space is a function $(\mathbf{x}, \mathbf{y}) \mapsto B(\mathbf{x}, \mathbf{y})$ (where $\mathbf{x}, \mathbf{y}$ is an arbitrary pair of vectors, and $B(\mathbf{x}, \mathbf{y})$ is a scalar) which in each of the inputs $\mathbf{x}, \mathbf{y}$ satisfies the *linearity property*: for all vectors $\mathbf{x}, \mathbf{x}', \mathbf{y}$ and scalars λ, λ':

$$B(\lambda\mathbf{x} + \lambda'\mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \lambda' B(\mathbf{x}', \mathbf{y}),$$
$$B(\mathbf{y}, \lambda\mathbf{x} + \lambda'\mathbf{x}') = \lambda B(\mathbf{y}, \mathbf{x}) + \lambda' B(\mathbf{y}, \mathbf{x}').$$

A bilinear form is called *symmetric* if $B(\mathbf{y}, \mathbf{x}) = B(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$.

1. Prove that the dot-product $\mathbf{x} \cdot \mathbf{y} = \sum x_i y_i$ is a symmetric bilinear form.

2. Prove that if Q is a quadratic form then

$$B_Q(\mathbf{x}, \mathbf{y}) := \frac{1}{2} \left[Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}) \right]$$

is a symmetric bilinear form.

3. Vice versa, show that if B is a bilinear form, then $Q_B(\mathbf{x}) := B(\mathbf{x}, \mathbf{x})$ is a quadratic form with coefficients $q_{ij} = \left[B(\mathbf{e}_i, \mathbf{e}_j) + B(\mathbf{e}_j, \mathbf{e}_i) \right] / 2$ (here $\mathbf{e}_i = (\dots, 0, 1, 0, \dots)$ is the ith coordinate vector), and that $Q \mapsto B_Q$ and $B \mapsto Q_B$ are inverse correspondences between quadratic forms and symmetric bilinear forms.

`Remark.` The correspondence between quadratic and symmetric bilinear forms remains true not only over $\mathbb{R}$ or $\mathbb{C}$, but over any field of scalars, $\mathbb{F}$, where division by 2 is well-defined (i.e. where $1 + 1 \neq 0$). The next result also remains valid in this generality.

4. Prove that for every symmetric bilinear form there exists a basis $\mathbf{f}_1, \dots, \mathbf{f}_n$ in which the coefficient matrix $[B(\mathbf{f}_i, \mathbf{f}_j)]$ is diagonal.

`Hint:` Construct inductively a basis such that $\mathbf{f}_i$ is B-orthogonal to all previous $\mathbf{f}_j$: $B(\mathbf{f}_i, \mathbf{f}_j) = 0$ for all $j < i$.

`Definition.` Two quadratic forms, Q and Q', on the same space are called *equivalent* if there exists a linear change of coordinates that transforms one into the other.

5. Prove that every quadratic form in $\mathbb{C}^n$ is equivalent to one of the normal forms: $z_1^2 + \cdots + z_r^2$, $r = 0, 1, \dots, n$.

6. Prove that every quadratic form in $\mathbb{R}^n$ is equivalent to one of the normal forms: $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, $0 \leq p + q \leq n$.

7. List all the six normal forms of quadratic forms in $\mathbb{R}^2$ and sketch their graphs. (Recall that the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the surface in $\mathbb{R}^3$ given by the equation $z = f(x, y)$.)

`Definition.` A quadratic form in $\mathbb{R}^n$ is called *positive definite* (*negative definite*) if its values are positive (resp. negative) everywhere outside the origin. The maximal dimension of subspaces on which a given quadratic form is positive (negative) definite is called the *positive* (resp. *negative*) *inertia index* of this quadratic form.

8. Prove that inertia indices of the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ are p and q respectively.
`Hint:` Prove that every subspace $W \subset \mathbb{R}^n$ of dimension $p+1$ contains a non-zero vector satisfying the equations: $x_1 = \cdots = x_p = 0$.

9. Prove that two quadratic forms in $\mathbb{R}^n$ are equivalent if and only if they have the same inertia indices.

10. Find a necessary and sufficient condition for two quadratic forms in $\mathbb{C}^n$ to be equivalent. `Answer:` The rank r of the coefficient matrix $[q_{ij}]$ is the only invariant.

11.* Let $\mathbf{a}_1, \ldots, \mathbf{a}_p$ and $\mathbf{b}_1, \ldots, \mathbf{b}_q$ be linear forms in $\mathbb{R}^n$, and let $Q(\mathbf{x}) = \mathbf{a}_1^2(\mathbf{x}) + \cdots + \mathbf{a}_p^2(\mathbf{x}) - \mathbf{b}_1^2(\mathbf{x}) - \cdots - \mathbf{b}_q^2(\mathbf{x})$. Prove that the positive and negative inertia indices of Q do not exceed p and q respectively.
`Hint:` Consider the map $\mathbb{R}^n \to \mathbb{R}^{p+q}$ defined by the linear forms.

12. Let B denote the matrix of coefficients $B(\mathbf{e}_i, \mathbf{e}_j)$ of a bilinear form, and let $\mathbf{x} = C\mathbf{x}'$ be a change of coordinates. Show that in new coordinates, the transformed bilinear form (i.e. $B(C\mathbf{x}', C\mathbf{y}')$) has coefficient matrix $B' = C^t BC$. (Here C^t denotes the matrix transposed to C.)

13. Classify real (complex) symmetric matrices B up to transformations $B \mapsto C^t BC$, where C is an invertible real (resp. complex) matrix.

14. Find orthogonal bases and inertia indices of quadratic forms:

$$x_1x_2 + x_2^2, \quad x_1^2 + 4x_1x_2 + 6x_2^2 - 12x_2x_3 + 18x_3^2, \quad x_1x_2 + x_2x_3 + x_3x_1.$$

15. Classify all quadratic curves $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$,$abc \neq 0$, on the real (complex) plane up to linear inhomogeneous changes of coordinates $x = \alpha x' + \beta y' + \lambda, y = \gamma x' + \delta y' + \mu$.
`Answer over` $\mathbb{C}$`:` There are five equivalence classes: circle, hyperbola, pairs of crossing lines, parallel lines, coinciding lines.

`Definition.` The determinant of the coefficient matrix of a quadratic form is often called the *discriminant* of the quadratic form. When the discriminant is non-zero, the quadratic form is called *non-degenerate*. In a $n \times n$-matrix, the $k \times k$-determinants in the left upper corner, $k = 1, 2, \ldots, n$, are called *leading principal minors*.

16. Show that a quadratic form is non-degenerate if and only if the corresponding symmetric bilinear form B is non-degenerate, i.e. if for every non-zero vector $\mathbf{x}$ there exists a vector $\mathbf{y}$ such that $B(\mathbf{x}, \mathbf{y}) \neq 0$.

17. How does the discriminant of a quadratic form Q change under a linear change of coordinates $\mathbf{x} = C\mathbf{x}'$?

18. Let Q be a quadratic form in $\mathbb{R}^n$ such that all its leading principal minors $\Delta_k \neq 0$. Prove *Sylvester's Rule*: The negative inertia index of Q is equal to the number of sign changes in the sequence of numbers $\Delta_0 = 1, \Delta_1, \ldots, \Delta_n$. `Hint:` Improve Problem 3 by finding $\mathbf{f}_k \in Span(\mathbf{e}_1, \ldots, \mathbf{e}_k)$.

19. Use Sylvester's Rule to find inertia indices of quadratic forms:

$$x_1^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_4, \quad x_1x_2 - x_2^2 + x_3^2 + 2x_2x_4 + x_4^2.$$

20.* Classify quadratic forms in $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ is the finite field of p elements. Consider separately the case of the prime $p > 2$ and $p = 2$.