Definitions. A quadratic form on the vector space \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)) is a homogeneous degree-2 polynomial \( Q(\mathbf{x}) = \sum_{i,j} q_{ij} x_i x_j \), where we may assume that \( q_{ij} = q_{ji} \) for all \( i, j \) and where \((x_1, \ldots, x_n)\) are coordinates of the vector \( \mathbf{x} \). A bilinear form on the vector space is a function \( (\mathbf{x}, \mathbf{y}) \mapsto B(\mathbf{x}, \mathbf{y}) \) (where \( \mathbf{x}, \mathbf{y} \) is an arbitrary pair of vectors, and \( B(\mathbf{x}, \mathbf{y}) \) is a scalar) which in each of the inputs \( \mathbf{x}, \mathbf{y} \) satisfies the linearity property: for all vectors \( \mathbf{x}, \mathbf{x}', \mathbf{y} \) and scalars \( \lambda, \lambda' \):

\[
B(\lambda \mathbf{x} + \lambda' \mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \lambda' B(\mathbf{x}', \mathbf{y}),
\]

\[
B(\mathbf{y}, \lambda \mathbf{x} + \lambda' \mathbf{x}') = \lambda B(\mathbf{y}, \mathbf{x}) + \lambda' B(\mathbf{y}, \mathbf{x}').
\]

A bilinear form is called symmetric if \( B(\mathbf{y}, \mathbf{x}) = B(\mathbf{x}, \mathbf{y}) \) for all \( \mathbf{x}, \mathbf{y} \).

1. Prove that the dot-product \( \mathbf{x} \cdot \mathbf{y} = \sum x_i y_i \) is a symmetric bilinear form.

2. Prove that if \( Q \) is a quadratic form then

\[
B_Q(\mathbf{x}, \mathbf{y}) := \frac{1}{2} [Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})]
\]

is a symmetric bilinear form.

3. Vice versa, show that if \( B \) is a bilinear form, then \( Q_B(\mathbf{x}) := B(\mathbf{x}, \mathbf{x}) \) is a quadratic form with coefficients \( q_{ij} = [B(e_i, e_j) + B(e_j, e_i)]/2 \) (here \( e_i = (\ldots, 0, 1, 0, \ldots) \) is the \( i \)th coordinate vector), and that \( Q \mapsto B_Q \) and \( B \mapsto Q_B \) are inverse correspondences between quadratic forms and symmetric bilinear forms.

Remark. The correspondence between quadratic and symmetric bilinear forms remains true not only over \( \mathbb{R} \) or \( \mathbb{C} \), but over any field of scalars, \( \mathbb{F} \), where division by 2 is well-defined (i.e. where \( 1 + 1 \neq 0 \)). The next result also remains valid in this generality.

4. Prove that for every symmetric bilinear form there exists a basis \( \mathbf{f}_1, \ldots, \mathbf{f}_n \) in which the coefficient matrix \([B(\mathbf{f}_i, \mathbf{f}_j)]\) is diagonal.

Hint: Construct inductively a basis such that \( \mathbf{f}_i \) is \( B \)-orthogonal to all previous \( \mathbf{f}_j \): \( B(\mathbf{f}_i, \mathbf{f}_j) = 0 \) for all \( j < i \).

Definition. Two quadratic forms, \( Q \) and \( Q' \), on the same space are called equivalent if there exists a linear change of coordinates that transforms one into the other.

5. Prove that every quadratic form in \( \mathbb{C}^n \) is equivalent to one of the normal forms: \( z_1^2 + \cdots + z_r^2, \ r = 0, 1, \ldots, n. \)

6. Prove that every quadratic form in \( \mathbb{R}^n \) is equivalent to one of the normal forms:

\[
x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, \ 0 \leq p + q \leq n.
\]

7. List all the six normal forms of quadratic forms in \( \mathbb{R}^2 \) and sketch their graphs. (Recall that the graph of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) is the surface in \( \mathbb{R}^3 \) given by the equation \( z = f(x, y) \).)

Definition. A quadratic form in \( \mathbb{R}^n \) is called positive definite (negative definite) if its values are positive (resp. negative) everywhere outside the origin. The maximal dimension of subspaces on which a given quadratic form is positive (negative) definite is called the positive (resp. negative) inertia index of this quadratic form.
8. Prove that inertia indices of the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ are $p$ and $q$ respectively.

**Hint:** Prove that every subspace $W \subset \mathbb{R}^n$ of dimension $p+1$ contains a non-zero vector satisfying the equations: $x_1 = \cdots = x_p = 0$.

9. Prove that two quadratic forms in $\mathbb{R}^n$ are equivalent if and only if they have the same inertia indices.

10. Find a necessary and sufficient condition for two quadratic forms in $\mathbb{C}^n$ to be equivalent. **Answer:** The rank $r$ of the coefficient matrix $[\alpha_{ij}]$ is the only invariant.

11. * Let $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ be linear forms in $\mathbb{R}^n$, and let $Q(x) = a_1^2(x) + \cdots + a_p^2(x) - b_1^2(x) - \cdots - b_q^2(x)$. Prove that the positive and negative inertia indices of $Q$ do not exceed $p$ and $q$ respectively.

**Hint:** Consider the map $\mathbb{R}^n \to \mathbb{R}^{p+q}$ defined by the linear forms.

12. Let $B$ denote the matrix of coefficients $B(e_i, e_j)$ of a bilinear form, and let $x = Cx'$ be a change of coordinates. Show that in new coordinates, the transformed bilinear form (i.e. $B(Cx', Cy')$) has coefficient matrix $B' = C^tBC$. (Here $C^t$ denotes the matrix transposed to $C$.)

13. Classify real (complex) symmetric matrices $B$ up to transformations $B \leftrightarrow C^tBC$, where $C$ is an invertible real (resp. complex) matrix.

14. Find orthogonal bases and inertia indices of quadratic forms:

   \begin{align*}
   x_1 x_2 + x_2^2, & \quad x_1^2 + 4 x_1 x_2 + 6 x_2^2 - 12 x_2 x_3 + 18 x_3^2, \quad x_1 x_2 + x_2 x_3 + x_3 x_1.
   \end{align*}

15. Classify all quadratic curves $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$, $abc \neq 0$, on the real (complex) plane up to linear inhomogeneous changes of coordinates $x = \alpha x' + \beta y' + \lambda, y = \gamma x' + \delta y' + \mu$.

**Answer over $\mathbb{C}$:** There are five equivalence classes: circle, hyperbola, pairs of crossing lines, parallel lines, coinciding lines.

**Definition.** The determinant of the coefficient matrix of a quadratic form is often called the **discriminant** of the quadratic form. When the discriminant is non-zero, the quadratic form is called **non-degenerate.** In a $n \times n$-matrix, the $k \times k$-determinants in the left upper corner, $k = 1, 2, \ldots, n$, are called **leading principal minors.**

16. Show that a quadratic form is non-degenerate if and only if the corresponding symmetric bilinear form $B$ is non-degenerate, i.e. if for every non-zero vector $x$ there exists a vector $y$ such that $B(x, y) \neq 0$.

17. How does the discriminant of a quadratic form $Q$ change under a linear change of coordinates $x = Cx'$?

18. Let $Q$ be a quadratic form in $\mathbb{R}^n$ such that all its leading principal minors $\Delta_k \neq 0$. Prove **Sylvester’s Rule:** The negative inertia index of $Q$ is equal to the number of sign changes in the sequence of numbers $\Delta_0 = 1, \Delta_1, \ldots, \Delta_n$. **Hint:** Improve Problem 3 by finding $f_i \in \text{Span}(e_1, \ldots, e_k)$.

19. Use Sylvester’s Rule to find inertia indices of quadratic forms:

   \begin{align*}
   x_1^2 + 2x_1 x_2 + 2 x_2 x_3 + 2 x_1 x_4, & \quad x_1 x_2 - x_2^2 + x_3^2 + 2 x_2 x_4 + x_4^2.
   \end{align*}

20. * Classify quadratic forms in $\mathbb{F}^n$, where $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ is the finite field of $p$ elements. Consider separately the case of the prime $p > 2$ and $p = 2$. 