LINEAR ALGEBRA. Part 0

`Definitions.` Let $\mathbb{F}$ stands for $\mathbb{R}$, or $\mathbb{C}$, or actually any field. We denote by $\mathbb{F}^n$ the set of all *n-vectors*, i.e. $n \times 1$-matrices with entries from $\mathbb{F}$. Equipped with the operations of addition and multiplication by scalars, they form an $\mathbb{F}$-vector space. A map $A : \mathbb{F}^n \to \mathbb{F}^m$ is called *linear*, if for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ and all $\lambda, \mu \in \mathbb{F}$, we have $A(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda A\mathbf{x} + \mu A\mathbf{y}$. Two $\mathbb{F}$-vector spaces are called *isomorphic* if there exists an invertible linear map between them. Two linear maps $A, B : \mathbb{F}^n \to \mathbb{F}^m$ are called *equivalent* if there exists isomorphisms $C : \mathbb{F}^m \to \mathbb{F}^m$ and $D : \mathbb{F}^n \to \mathbb{F}^n$ such that $B = C^{-1}AD$. The *dimension* of a vector space is defined as the maximal cardinality of linearly independent subsets in it The *rank* of a linear map is defined as the dimension of its range (which is a subspace in the target space, and is therefore a vector space on its own).

1. Show that every linear map $\mathbb{F}^n \to \mathbb{F}^m$ is the multiplication by an $m \times n$-matrix, A: $\mathbf{x} \mapsto A\mathbf{x}$.

2. Prove that in $\mathbb{F}^n$, every set of $n + 1$ vectors are linearly dependent.
`Hint:` Apply induction on n.

3. Prove that every maximal linearly independent set in $\mathbb{F}^n$ has n elements.
`Hint:` Show first that vectors of this set form a *basis*, i.e. every vector is written uniquely as their linear combination.

4. (*Classification of finite dimensional vector spaces.*) Prove that every finite dimensional $\mathbb{F}$-vector space is isomorphic to exactly one of $\mathbb{F}^n$, $n = 0, 1, 2, \dots$.

5. (*Classification of linear maps: The Rank Theorem.*) Prove that two linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$ are equivalent if and only if they have the same rank.
`Hint:` Given A of rank r, construct bases $\mathbf{e}_1, \dots, \mathbf{e}_n$ in $\mathbb{F}^n$ and $\mathbf{f}_1, \dots, \mathbf{f}_m$ in $\mathbb{F}^m$ such that $A\mathbf{e}_i = \mathbf{f}_i$ for $i = 1, \dots r$, and $A\mathbf{e}_i = \mathbf{0}$ for $i > r$.

6. Derive that every linear map $\mathbb{F}^n \to \mathbb{F}^m$ of rank r is equivalent to the map given by the $m \times n$-matrix $E_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

7. Derive that for every linear map $A : \mathbb{F}^n \to \mathbb{F}^m$, $\text{rk}(A) + \text{nullity}(A) = n$.

8. Show that for every $m \times n$-matrix of rank r there exist invertible matrices C and D such that $A = C^{-1}E_rD$.

9. (*Systems of linear equations: theory.*) A system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns with the coefficient matrix A of rank r is consistent provided that the right hand side $\mathbf{b}$ satisfies a certain set of r linear condidtion, and in this case the general solution depends on $n - r$ parameters.
`Hint:` This is true for the system $E_r\mathbf{x} = \mathbf{b}$.

10. (*Excess dimension formula.*) Let U and V be two subspaces in $\mathbb{F}^n$ of dimensions a and b respectively. If $U + V = \mathbb{F}^n$ (i.e. vectors from U and V span the whole space), then $\dim U \cap V = n - a - b$. `Hint:` Consider the map $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} - \mathbf{v}$ to $\mathbb{F}^n$ from the *direct sum* $U \oplus V$ (by definition, it consists of ordered pairs, $\mathbf{u} \in U, \mathbf{v} \in V$), and apply the "rank+nullity" formula.

11. Prove that positive and negative inertia indices of the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots x_{p+q}^2$ in n real variables $x_1, \dots, x_n$ are equal to p and q respectively.