PERMUTATIONS AND DETERMINANTS

Definition. A permutation on a set $S$ is an invertible function from $S$ to itself.

1. Prove that permutations on $S$ form a group with respect to the operation of composition, i.e., that (i) composition of permutations is a permutation, (ii) the operation is associative: $(fg)h = f(gh)$ for all permutations $f, g, h$, (iii) there exists the identity permutation $id$ such that $idf = f$ and $f id = f$ for every $f$, and (iv) every permutation has its inverse: $ff^{-1} = f^{-1}f = id$.

2. Prove that on the set $S = \{1, \ldots, n\}$, there are $n!$ permutations.

Remark. A permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is usually described by the table $\sigma(i_1) = i_1, \ldots, \sigma(i_n) = i_n$, where $i_k = \sigma(k)$. The finite group formed by $n!$ of such permutations is denoted by $S_n$.

3. List all elements of $S_n$ for $n = 1, 2, 3, 4$.

Definition. A permutation $\sigma = (i_1, \ldots, i_n)$ acts on polynomials $P$ in $n$ variables $\{x_1, \ldots, x_n\}$ by the rule $(\sigma P)(x_1, \ldots, x_n) := P(x_{i_1}, \ldots, x_{i_n})$.

4. Take $P = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ and prove that $\sigma P = \pm P$, where $\sigma = \pm 1$ for every permutation $\sigma \in S_n$. Prove that $\text{even } \sigma \mapsto \text{even } \sigma$ and $\text{odd } \sigma \mapsto \text{odd } \sigma$.

Remark. The last formula means that $\sigma$ is a homomorphism of the group $S_n$ to the group consisting of two numbers $\{1, -1\}$. Permutations with the sign $\epsilon = 1$ are called even, and those with $\epsilon = -1$ odd.

5. The length $l(\sigma)$ of a permutation $\sigma$ is defined as the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$ (they are called “pairs in inversion”). Prove that $l(\sigma) = (-1)^{l(\sigma)}$.

6. List all even and all odd permutations of $S_4$ with $n = 1, 2, 3, 4$.

7. Prove that the composition of two even (odd) permutations is even, and of even and odd is odd, and that there are exactly $n!/2$ even (odd) permutations on the set $\{1, \ldots, n\}$.

Definition. A permutation swapping two indices, $i$ and $j$, and leaving all other indices unchanged is called a transposition and denoted $\tau_{ij}$.

8. Prove that every permutation $\sigma$ can be written (non-uniquely) as a composition of transpositions, $\sigma = \tau_1 \cdots \tau_N$, and that $\epsilon(\sigma) = (-1)^N$.

Hint: Show that under composition of $\sigma$ with $\tau_{ij}$, the length decreases when the pair $i, j$ is in inversion and increases when it is not.

Definition. The determinant of $n \times n$-matrix $A$ is defined as

$$\det A := \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

9. Prove that $\det A^t = \det A$.

10. Let $A = [a_{ij}]$ represents a square matrix $A$ written “by columns.” Prove the following properties of determinants:

(i) Transposing two columns (rows) of $A$ changes the sign of the determinant: $\det[\ldots a_j \ldots a_i \ldots] = - \det[\ldots a_i \ldots a_j \ldots]$.

(ii) If a column (row) of $A$ is multiplied by a scalar $\lambda$, then the determinant increases $\lambda$ times, e.g., $\det[\lambda a_1, \ldots] = \lambda \det[a_1, \ldots]$. 

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The determinant does not change if a multiple of one column (row) is added to another one.

Remark. Property (i) means that the det as a function of columns of a matrix is \textit{totally antisymmetric}, i.e. under a permutation of columns it changes the sign according to the parity of the permutation. Column properties (ii) and (iii) together are equivalent to the following \textit{poly-linearity} property, i.e. \textit{linearity} with respect to each column: for all scalars \( \lambda \) and \( \mu \),

\[
\det[\ldots, \lambda \mathbf{a} + \mu \mathbf{b}, \ldots] = \lambda \det[\ldots, \mathbf{a}, \ldots] + \mu \det[\ldots, \mathbf{b}, \ldots].
\]

10. Prove that every function of \( n \) columns of size \( n \) which is totally antisymmetric and is linear with respect to each of them is proportional to the determinant function of the matrix formed from these columns.

11. Prove that \( \det(AB) = (\det A)(\det B) \).

\textbf{Hint:} Show that the function \((\mathbf{b}_1, \ldots, \mathbf{b}_n) \mapsto \det[A\mathbf{b}_1, \ldots, A\mathbf{b}_n]\) is poly-linear and totally antisymmetric, and is thus proportional to \( \det[\mathbf{b}_1, \ldots, \mathbf{b}_n] \).

12. Prove that if \( A \) is invertible (over a ring of scalars \( R \)), then \( \det A \) is invertible (in \( R \)).

\textbf{Remark.} Invertible \( n \times n \)-matrices (over a commutative ring \( R \) with unity) form a group w.r.t. matrix product. It is denoted \( GL_n(R) \) and called the \textit{general linear group}. Problems 11 and 12 show that \( \det \) is a homomorphism of groups: \( GL_n(R) \to R^* \).

\textbf{Definition.} The cofactor \( C_{ij} \) of a square matrix \( A \) is the scalar that differs by the sign \((-1)^{i+j}\) from the determinant obtained from the matrix \( A \) by removing the \( i \)th row and \( j \)th column.

13. Prove the cofactor expansion formulas:

\[
\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} C_{ij} a_{ij}.
\]

\textbf{Hint:} In the definition of \( \det A \), pull out the factors \( a_{ij} \) with a fixed \( i \) (or fixed \( j \)).

14. Prove that if \( i' \neq i \) (resp. \( j' \neq j \)), then

\[
\sum_{j=1}^{n} a_{ij} C_{ij} = 0, \quad \sum_{i=1}^{n} C_{ij} a_{ij'} = 0.
\]

\textbf{Hint:} Corrupt the matrix \( A \) by replicating the \( i' \)th row in place of the \( i \)th one.

15. Let \( C \) denote the matrix \([C_{ij}]\) of cofactors of a matrix \( A \). Prove that \( AC^t = (\det A)I = C^t A \). Deduce (for a matrix \( A \) with entries from \( R \)) that if \( \det A \) is invertible (in \( R \)) then the matrix \( A \) is invertible (over \( R \), i.e. there exists a matrix \( A^{-1} \) with entries from \( R \) such that \( AA^{-1} = I = A^{-1}A \)).