PERMUTATIONS AND DETERMINANTS

`Definition.` A *permutation* on a set S is an invertible function from S to itself.

1. Prove that permutations on S form a *group* with respect to the operation of composition, i.e. that (i) composition of permutations is a permutation, (ii) the operation is associative: $(fg)h = f(gh)$ for all permutations f, g, h, (iii) there exists the identity permutation id such that $\mathrm{id}\, f = f\, \mathrm{id} = f$ for every f, and (iv) every permutation has its inverse: $ff^{-1} = f^{-1}f = \mathrm{id}$.

2. Prove that on the set $S = \{1, \dots, n\}$, there are $n!$ permutations.

`Remark.` A permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is usually described by the table $\binom{1,\dots,n}{i_1,\dots,i_n}$. where $i_k = \sigma(k)$. The finite group formed by $n!$ of such permutations is denoted by S_n.

3. List all elements of S_n for $n = 1, 2, 3, 4$.

`Definition.` A permutation $\sigma = \binom{1,\dots,n}{i_1,\dots,i_n}$ *acts* on polynomials P in n variables $(x_1, \dots, x_n)$ by the rule $(\sigma P)(x_1, \dots, x_n) := P(x_{i_1}, \dots, x_{i_n})$.

4. Take $P = \prod_{1 \le i < j \le n}(x_i - x_j)$ and prove that $\sigma P = \epsilon(\sigma)P$, where $\epsilon(\sigma) = \pm 1$ for every permutation $\sigma \in S_n$. Prove that $\epsilon(\sigma\sigma') = \epsilon(\sigma)\epsilon(\sigma')$.

`Remark.` The last formula means that ϵ is a homomorphism of the group S_n to the group consisting of two numbers $\{1, -1\}$. Permutations with the sign $\epsilon = 1$ are called *even*, and those with $\epsilon = -1$ *odd*.

5. The *length* $l(\sigma)$ of a permutation σ is defined as the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$ (they are called "pairs in inversion"). Prove that $\epsilon(\sigma) = (-1)^{l(\sigma)}$.

6. List all even and all odd permutations of S_n with $n = 1, 2, 3, 4$.

7. Prove that the composition of two even (odd) permutations is even, and of even and odd is odd, and that there are exactly $n!/2$ even (odd) permutations on the set $\{1, \dots, n\}$.

`Definition.` A permutation swapping two indices, i and j, and leaving all other indices unchanged is called a *transposition* and denoted τ_{ij}.

8. Prove that every permutation σ can be written (non-uniquely) as a composition of transpositions, $\sigma = \tau_1 \cdots \tau_N$, and that $\epsilon(\sigma) = (-1)^N$.
`Hint:` Show that under composition of σ with τ_{ij}, the length decreases when the pair i, j is in inversion and increases when it is not.

`Definition.` The *determinant* of $n \times n$-matrix A is defined as

$$\det A := \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

9. Prove that $\det A^t = \det A$.

10. Let $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ represents a square matrix A written "by columns." Prove the following properties of determinants:

(i) Transposing two columns (rows) of A changes the sign of the determinant: $\det[\dots \mathbf{a}_j \dots \mathbf{a}_i \dots] = -\det[\dots \mathbf{a}_i \dots \mathbf{a}_j \dots]$.

(ii) If a column (row) of A is multiplied by a scalar λ, then the determinant increases λ times, e.g. $\det[\lambda \mathbf{a}_1, \dots] = \lambda \det[\mathbf{a}_1, \dots]$.

(iii) The determinant does not change if a multiple of one column (row) is added to another one.

(iv) $\det I = 1$.

`Remark.` Property (i) means that the det as a function of columns of a matrix is *totally antisymmetric*, i.e. under a permutation of columns it changes the sign according to the parity of the permutation. Column properties (ii) and (iii) together are equivalent to the following *poly-linearity* property, i.e. *linearity* with respect to each column: for all scalars λ and μ,

$$\det[\dots, \lambda \mathbf{a} + \mu \mathbf{b}, \dots] = \lambda \det[\dots, \mathbf{a}, \dots] + \mu \det[\dots, \mathbf{b}, \dots].$$

10. Prove that every function of n columns of size n which is totally antisymmetric and is linear with respect to each of them is proportional to the determinant function of the matrix formed from these columns.

11. Prove that $\det(AB) = (\det A)(\det B)$.

`Hint:` Show that the function $(\mathbf{b}_1, \dots, \mathbf{b}_n) \mapsto \det[A\mathbf{b}_1, \dots, A\mathbf{b}_n]$ is poly-linear and totally antisymmetric, and is thus proportional to $\det[\mathbf{b}_1, \dots, \mathbf{b}_n]$.

12. Prove that if A is invertible (over a ring of scalars R), then $\det A$ is invertible (in R).

`Remark.` Invertible $n \times n$-matrices (over a commutative ring R with unity) form a group w.r.t. matrix product. It is denoted $GL_n(R)$ and called the *general linear* group. Problems 11 and 12 show that det is a homomorphism of groups: $GL_n(R) \to R^*$.

`Definition.` The cofactor C_{ij} of a square matrix A is the scalar that differs by the sign $(-1)^{i+j}$ from the determinant obtained from the matrix A by removing the ith row and jth column.

13. Prove the cofactor expansion formulas:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} C_{ij} a_{ij}.$$

`Hint:` In the definition of $\det A$, pull out the factors a_{ij} with a fixed i (or fixed j).

14. Prove that if $i' \neq i$ (resp. $j' \neq j$), then

$$\sum_{j=1}^{n} a_{i'j} C_{ij} = 0, \quad \sum_{i=1}^{n} C_{ij} a_{ij'} = 0.$$

`Hint:` Corrupt the matrix A by replicating the i'th row in place of the ith one.

15. Let C denote the matrix $[C_{ij}]$ of cofactors of a matrix A. Prove that $AC^t = (\det A)I = C^t A$. Deduce (for a matrix A with entries from R) that if $\det A$ is invertible (in R) then the matrix A is invertible (over R, i.e. there exists a matrix A^{-1} with entries from R such that $AA^{-1} = I = A^{-1}A$).