EXERCISES ON DETERMINANTS

1. Prove that the following determinant is equal to 0:

$$\begin{vmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ v & w & x & y & z \end{vmatrix}.$$

2. Compute determinants:

$$\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}, \quad \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix}, \quad \begin{vmatrix} \cos x & \sin y \\ \sin x & \cos y \end{vmatrix}.$$

3. Compute determinants:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{vmatrix}.$$

4. For each of the 24 permutations of $\{1,2,3,4\}$, find the length and sign.
5. Find the length of the following permutation:

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & 2k \\ 1 & 3 & \dots & 2k-1 & 2 & 4 & \dots & 2k \end{pmatrix}.$$

6. Find the maximal possible length of permutations of $\{1, ..., n\}$.
7. Find the length of a permutation $\begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix}$ given the length l of the permutation $\begin{pmatrix} 1 & \dots & n \\ i_n & \dots & i_1 \end{pmatrix}$.
8. Prove that inverse permutations have the same length.
9. Compare parities of permutations of the letters a,g,h,i,l,m,o,r,t in the words *logarithm* and *algorithm*.
10. Represent the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$ as composition of a minimal number of transpositions.
11. Do products $a_{13}a_{24}a_{53}a_{41}a_{35}$ and $a_{21}a_{13}a_{34}a_{55}a_{42}$ occur in the defining formula for determinants of size 5?
12. Find the signs of the elementary products $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$ and $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$ in the definition of determinants of size 6 by computing the numbers of inverted pairs of indices.
13. Compute the determinants

$$\begin{vmatrix} 13247 & 13347 \\ 28469 & 28569 \end{vmatrix}, \quad \begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix}.$$

14. The numbers 195, 247, and 403 are divisible by 13. Prove that the following determinant is also divisible by 13: $\begin{vmatrix} 1 & 9 & 5 \\ 2 & 4 & 7 \\ 4 & 0 & 3 \end{vmatrix}$.

15. Professor Dumbel writes his office and home phone numbers as a 7×1-matrix O and 1×7-matrix H respectively. Help him compute $\det(OH)$.

16. How does a determinant change if all its n columns are rewritten in the opposite order?

17.*** Solve the equation $\begin{vmatrix} 1 & x & x^2 & ... & x^n \\ 1 & a_1 & a_1^2 & ... & a_1^n \\ 1 & a_2 & a_2^2 & ... & a_2^n \\ & & & ... & \\ 1 & a_n & a_n^2 & ... & a_n^n \end{vmatrix} = 0$, where all $a_1, ..., a_n$ are given distinct numbers.

18. Prove that an anti-symmetric matrix of size n has zero determinant if n is odd.

19. How do similarity transformations of a given matrix affect its determinant?

`Definition.` Given a square matrix A, the matrix $[C_{ij}]^T$ transposed to the matric formed by cofactors of A is (often) called the matrix *adjoint* to C and denoted $\text{adj}(A)$.

20. Prove that the adjoint matrix of an upper (lower) triangular matrix is upper (lower) triangular.

21. Which triangular matrices are invertible?

22. Compute the determinants: ($*$ is a wild card):

$$(a) \begin{vmatrix} * & * & * & a_n \\ * & * & \dots & 0 \\ * & a_2 & 0 & \dots \\ a_1 & 0 & \dots & 0 \end{vmatrix}, \quad (b) \begin{vmatrix} * & * & a & b \\ * & * & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{vmatrix}.$$

23. Compute determinants using cofactor expansions:

$$(a) \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix}, \quad (b) \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}.$$

24. Compute inverses of matrices using cofactor expansions:

$$(a) \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad (b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

25. Compute

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

26. Express $\det(\mathrm{adj}(A))$ of the adjoint matrix via $\det A$.
27. Which integer matrices have integer inverses?
28.* In the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, assume that D^{-1} exists and prove that $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\det D$.
29.* Compute determinants:

$$(a) \begin{vmatrix} 0 & x_1 & x_2 & \dots & x_n \\ x_1 & 1 & 0 & \dots & 0 \\ x_2 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ x_n & 0 & \dots & 0 & 1 \end{vmatrix}, \quad (b) \begin{vmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & d \end{vmatrix}.$$

`Definition.` By a *multi-index* I of length $|I| = k$ we mean an increasing sequence $i_1 < \dots < i_k$ of k indices from the set $\{1, \dots, n\}$. Given and $n \times n$-matrix A and two multi-indices I, J of the same length k, we define the (IJ)-*minor* of A as the determinant of the $k \times k$-matrix formed by the entries $a_{\mathbf{i}_\alpha j_\beta}$ of A located at the intersections of the rows $i_1, \dots, i_k$ with columns $j_1, \dots, j_k$ (see Figure 24). Also, denote by $\bar{I}$ the multi-index *complementary* to I, i.e. formed by those $n - k$ indices from $\{1, \dots, n\}$ which are *not* contained in I. Lagrange's formula below generalizes cofactor expansions.
30.* Prove that for each multi-index $I = (i_1, \dots, i_k)$, the following cofactor expansion with respect to rows $i_1, \dots, i_k$ holds true:

$$\det A = \sum_{J:|J|=k} (-1)^{i_1+\dots+i_k+j_1+\dots+j_k} M_{IJ} M_{\bar{I}\bar{J}},$$

where the sum is taken over all multi-indices $J = (j_1, \dots, j_k)$ of length k. Formulate and prove the analogous statement for columns.
31.* Let P_{ij}, $1 \le i < j \le 4$, denote the 2×2-minor of a 2×4-matrix formed by the columns i and j. Prove the following *Plücker identity*

$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0.$$

32.* Let A and B be $k \times n$ and $n \times k$ matrices (think of $k < n$). For each multi-index $I = (i_1, \dots, i_k)$, denote by A_I and B_I the $k \times k$-matrices formed by respectively: columns of A and rows of B with the indices $i_1, \dots, i_k$. Prove that the determinant of the $k \times k$-matrix AB is given by the following

Binet–Cauchy formula:
$$\det AB = \sum_I (\det A_I)(\det B_I).$$

33. The *cross product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is defined by
$$\mathbf{x} \times \mathbf{y} := \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$
Prove that the length $|\mathbf{x} \times \mathbf{y}| = \sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y} \rangle^2}$.

34.* Prove that $a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\cdots + \cfrac{1}{a_1 + \cfrac{1}{a_0}}}} = \dfrac{\Delta_n}{\Delta_{n-1}}$,

where $\Delta_n = \begin{vmatrix} a_0 & 1 & 0 & \dots & 0 \\ -1 & a_1 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & \dots & -1 & a_{n-1} & 1 \\ 0 & \dots & 0 & -1 & a_n \end{vmatrix}$.

35.* Compute: $\begin{vmatrix} \lambda & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \dots & 0 \\ . & . & . & . & . \\ 0 & \dots & 0 & \lambda & -1 \\ a_n & a_{n-1} & \dots & a_2 & \lambda + a_1 \end{vmatrix}$.

36.* Compute: $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \binom{2}{1} & \binom{3}{1} & \dots & \binom{n}{1} \\ 1 & \binom{3}{2} & \binom{4}{2} & \dots & \binom{n+1}{2} \\ . & . & . & . & . \\ 1 & \binom{n}{n-1} & \binom{n+1}{n-1} & \dots & \binom{2n-2}{n-1} \end{vmatrix}$.

37.* Prove *Vandermonde's identity*
$$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ . & . & . & . & . \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

38.* Compute: $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^3 & 3^3 & \dots & n^3 \\ . & . & . & . & . \\ 1 & 2^{2n-1} & 3^{2n-1} & \dots & n^{2n-1} \end{vmatrix}$.