Putnam 1986 Solutions

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A1. The inequality $x^4+36 \le 13x^2$ factors as $(x+3)(x+2)(x-2)(x-3) \le 0$, i.e. $-3 \le x \le -2$ or $2 \le x \le 3$. We compute $f'(x) = 3(x+1)(x-1)$, so f is increasing on both these intervals. Since $f(-2) = -2$ and $f(3) = 18$, 18 is the maximum.

A2.

$$X = \frac{10^{20000}}{10^{100}+3} = \frac{10^{19900}}{1+\frac{3}{10^{100}}} = \sum_{i=0}^{\infty} (-3)^i 10^{19900-100i} \approx \sum_{i=0}^{199} (-3)^i 10^{19900-100i} = Y.$$

The finite sum Y is a positive integer, and all of its terms but the last are divisible by 10: $Y \equiv (-3)^{199} \equiv 3 \pmod{10}$. Since X is an alternating series, the difference $X - Y$ has the same sign and smaller magnitude than the first omitted term $(-3)^{200}10^{-100} = (9/10)^{100}$, which is positive and less than 1. Consequently $\lfloor X \rfloor = Y$, and the rightmost digit of $\lfloor X \rfloor$ is 3.

A3. The identity $\operatorname{Arccot}(n^2+n+1) = \operatorname{Arccot}(n) - \operatorname{Arccot}(n+1)$ is provable from the addition formula for tangent when $n > 0$ and becomes the identity $\pi/4 = \pi/2 - \pi/4$ when $n = 0$. Thus $\sum_{n=0}^{N} \operatorname{Arccot}(n^2+n+1) = \operatorname{Arccot}(0) - \operatorname{Arccot}(N+1)$, which as $N \to \infty$ tends to $\pi/2 - 0 = \pi/2$.

A4. It is not hard to see, by completing the pairs $(\alpha_{1,1}, \alpha_{i,j})$ and $(\alpha_{i,1}, \alpha_{1,j})$ to a transversal in the same way when $i, j \ne 1$, that all transversals of a matrix $A = [\alpha_{i,j}]$ have the same sum if and only if $\alpha_{i,j} = \alpha_{i,1} + \alpha_{1,j} - \alpha_{1,1}$ for all i and j. For any matrix A of the given type, let $X = \max_{1 \le i,j \le n}(\alpha_{1,i} - \alpha_{1,j})$ and $Y = \max_{1 \le k,m \le n}(\alpha_{k,1} - \alpha_{m,1})$. Note that $X + Y \le 2$, since $(\alpha_{1,i} - \alpha_{1,j}) + (\alpha_{k,1} - \alpha_{m,1}) = \alpha_{k,i} - \alpha_{m,j} \le 1 - (-1) = 2$. Using inclusion-exclusion to compute the number of matrices having each of the six possible pairs (X, Y), we get:

(X,Y)	Matrices
$(0,0)$	1
$(0,1)$	$2 \cdot 2^n - 4$
$(1,0)$	$2 \cdot 2^n - 4$
$(0,2)$	$3^n - 2 \cdot 2^n + 1$
$(2,0)$	$3^n - 2 \cdot 2^n + 1$
$(1,1)$	$4^n - 4 \cdot 2^n + 4$
Total	$4^n + 2 \cdot 3^n - 4 \cdot 2^n - 1.$

A5. Let $h_i(x) = \sum_{j<i} c_{ij} x_j - f_i(x)$. Plugging into the given condition easily yields $\partial h_i/\partial x_j = \partial h_j/\partial x_i$ for all i and j, so the h_i are the partial derivatives of some function g. Note that $f_i + \partial g/\partial x_i = f_i + h_i = \sum_{j<i} c_{ij} x_j$ is linear.

A6. We differentiate the given equality k times, $k = 0, 1, \ldots, n$, divide through by $k!$, and then plug in $x = 1$ to get

$$\begin{aligned} \sum_i a_i &= -1 \\ \sum_i a_i b_i &= 0 \\ \sum_i a_i \binom{b_i}{2} &= 0 \\ &\cdots \\ \sum_i a_i \binom{b_i}{n-1} &= 0 \\ \sum_i a_i \binom{b_i}{n} &= (-1)^n f(1). \end{aligned}$$

Let $P(x) = (x - b_1) \cdots (x - b_n)$. We can uniquely express $P(x) = \sum_{k=0}^{n} c_k \binom{x}{k}$; plugging in $x = 0$ yields $c_0 = (-1)^n \prod b_i$, while considering the leading term gives $c_n = n!$. Linearly combining the above equations with coefficients $c_0, \ldots, c_n$ gives $0 = \sum a_i P(b_i) = (-1)^{n+1} \prod b_i + (-1)^n n! f(1)$, so $f(1) = \prod b_i / n!$.

B1. Since they have the same area and the same base, the triangle is twice as high as the rectangle. The diameter of the circle through the apex of the triangle is thus cut by the sides of the rectangle in the ratio $2:1:2$, and so $h = 2/5$.

B2. Answer: $(0,0,0), (1,0,-1), (0,-1,1), (-1,1,0)$. Let $x - y = u$, $y - z = v$, and $z - x = w$. Moving all terms of the first equation to the left side and factoring gives $(x-y)(x+y+1-2z) = 0$, i.e. $u = 0$ or $w - v = 1$. Similarly, the second equation reads $v = 0$ or $u - w = 1$. Finally, the triples (u, v, w) corresponding to triples (x, y, z) are exactly those satisfying $u + v + w = 0$. Solving the systems of three linear equations corresponding to each of the four cases yields the four solutions above.

B3. We induct on n with the base case $n = 1$ already given. Let F_n and G_n be the polynomials in the induction hypothesis and let $u = h - F_n G_n \equiv 0 \pmod{p^n}$. Let $F_{n+1} = F_n + us$ and $G_{n+1} = G_n + ur$, noting that $F_{n+1} \equiv F_n \equiv f \pmod{p}$ and similarly for G_{n+1}. We get

$$F_{n+1} G_{n+1} = F_n G_n + u(rF_n + sG_n) + u^2 = F_n G_n + u + u(rF_n + sG_n - 1) + u^2.$$

The last term clearly drops out $\pmod{p^{n+1}}$ since it is divisible by p^{2n}. Also, the next-to-last term is the product of a multiple of p^n and a multiple of p, so it is also $0 \pmod{p^{n+1}}$. We are left with $F_n G_n + u = h$, as desired.

B4. The assertion is true. Let M be a fixed positive integer and $M \le r < M+1$; we prove $0 \le G(r) \le 3/\sqrt{M}$, which clearly goes to 0 as $M \to \infty$. Specifically, we prove that $\sqrt{M^2 + 2(n+1)^2} - \sqrt{M^2 + 2n^2} < 3/\sqrt{M}$ whenever n is an integer with $M \le \sqrt{M^2 + 2n^2} < M + 1$, so that every r is within $3/\sqrt{M}$ of some $\sqrt{M^2 + 2n^2}$. To do this, we first solve the condition $\sqrt{M^2 + 2n^2} < M + 1$ for n, yielding $0 \le n < \sqrt{M + 1/2}$, i.e. $0 \le n \le \sqrt{M}$. Then rationalizing the numerator yields

$$\sqrt{M^2 + 2(n+1)^2} - \sqrt{M^2 + 2n^2} = \frac{4n + 2}{\sqrt{M^2 + 2(n+1)^2} + \sqrt{M^2 + 2n^2}} < \frac{2n+1}{M} \le \frac{2}{\sqrt{M}} + \frac{1}{M} \le \frac{3}{\sqrt{M}}.$$

B5. The assertion is false, as the example $p(x, y, z) = x$, $q(x, y, z) = y$, $r(x, y, z) = xy - z$ shows.

B6. Let $W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. The conditions of the problem reduce to $WSW^T S = -I$, where $S = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$. This says that WS and $-W^T S$ are inverses, which is equivalent to $W^T SWS = -I$. Expanding this in A, B, C, D and examining the upper left $n \times n$ submatrix yields $A^T D - B^T C = I$.