Putnam-1985. Solutions

A1. A triple (A_1, A_2, A_3) is uniquely determined by partitioning $\{1, \ldots, 10\}$ into 6 disjoint subsets $A_1 \cap A_2, A_2 \cap A_3, A_3 \cap A_1$, $A_1 - A_2 - A_3, A_2 - A_1 - A_3, A_3 - A_1 - A_2$. There are $6^{10} = 2^{10}3^{10}$ ways of coloring 10 items into 6 colors.

A2. Take base of T for the unit 1, and let y and x, $0 < y < x < 1$, be bases of rectangles S and R, parallel to the base of T. Then $A(S)/A(T) = 2y(x-y)$ and $A(R)/A(T) = 2x(1-x)$. For a fixed x, $2y(x-y)$ achieves maximum value $x^2/2$ at $y = x/2$. The sum $x^2/2 + 2x(1-x) = (4x - 3x^2)/2$ achieves maximum value $2/3$ at $x = 2/3$. Thus the maximum value of $(A(S) + A(R))/A(T) = 2/3$.

A3. $a_m(j+1) + 1 = (a_m(j) + 1)^2$, and hence $a_n(n) + 1 = (a_n(0) + 1)^{2^n} = (1 + d/2^n)^{2^n} \to e^d$ as $n \to \infty$. Thus $\lim_{n\to\infty} a_n(n) = e^d - 1$.

A4. $3^{20} \equiv 1 \bmod 100$ since $\phi(100) = \phi(4)\phi(25) = 2 \times 20$. Since $3^4 \equiv 1 \bmod 20$, we have $3^{3^3} \equiv 3^3 \bmod 20$, and hence $3^{3^{3^3}} \equiv 3^{3^3} \equiv 87 \bmod 100$. Thus, for $i \geq 3$, $a_i \equiv 87 \bmod 100$.

A5. $\int_0^{2\pi} e^{imx} dx = 0$ for $m \neq 0$. Thus we need to find for which $m = 1, ..., 10$, $(t+t^{-1}) \cdots (t^m + t^{-m})$ has a non-zero (and hence positive) coefficient at t^0, or equivalently, for which m $\{1, \ldots, m\}$ can be split into two groups of the same sum. For $m = 1, 2, 5, 6, 9, 10$, $1 + \cdots + m$ is odd, but $m = 3, 4, 7, 8$ will do: $1 + 2 = 3, 1 + 4 = 2 + 3, 7 + 6 + 1 = 2 + 3 + 4 + 5, 1 + 8 + 2 + 7 = 3 + 6 + 4 + 5$.

A6. For $p = a_0 + \cdots + a_m x^m$ with $a_0, a_m \neq 0$, put $p^* = a_m + \cdots + a_0 x^m$. Then $\Gamma(p) = \mathrm{Res}_{x=0}\, p(x)p^*(x)x^{-m-1}dx$. We have $f(x)f^*(x) = (3x + 1)(x+2)(3+x)(1+2x) = g(x)g^*(x)$ where $g = (3x+1)(1+2x)$ has $g(0) = 1$. Since $(p^n)^* = (p^*)^n$, we will have $\Gamma(g^n) = \Gamma(f^n)$ for all n.

B1. $(x-0)(x-1)(x+1)(x-2)(x+2) = x(x^2-1)(x^2-4) = x^5 - 5x^3 + 4x$ has $k = 3$ non-zero coefficients. A degree 5 polynomial p with 2 non-zero coefficients and no multiple roots will have the form $x^5 - ax$ or $x^5 - a$, and hence inevitably have nonreal roots (since $x^m = a$ has no more than 2 real solutions).

B2. Put $g_n(x) := x(x+n)^{n-1}$ and prove by induction on $n \geq 1$ that $f_n = g_n$. For $n = 1$, we obviously have $f_1 = x = g_1$. Assuming $f_n = g_n$, we find: $g'(x) = (x+n+1)^n + nx(x+n+1)^{n-1} = (n+1)(x+1)(x+n+1)^{n-1} = (n+1)f_n(x+1)$. Since $g_{n+1}(0) = 0$, we conclude $f_{n+1} = g_{n+1}$. Thus, $f_{100}(1) = 101^{99}$, where 101 is prime.

B3. Suppose that $a_{m,n} \leq mn$ for all $m, n \geq 1$. Denore $N(C)$ the number of pairs (m, n) with $mn < C$. Since $C^{-1}\int_1^C Cx^{-1}dx \to \infty$ as $C \to \infty$, $N(C)/C \to \infty$ too, and hence $N(C) > 8C$ for C large enough. Then among $N(C)$ terms $a_{m,n}$ with $mn < C$ some of the values $1, \ldots, [C]$ must occur more than 8 times.

B4. For a required rectangle to lie inside the circle, q must lie in the rectangle with vertices $(\pm x, \pm y)$ where $(x, y) = (\cos\alpha, \sin\alpha) = p$. The probability of this is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\pi}|2\cos\alpha|\ |2\sin\alpha| d\alpha = \frac{8}{\pi^2}\int_0^{\pi/2}\sin\alpha d\sin\alpha = \frac{4}{\pi^2}.$$

B5. $\int_0^\infty \frac{1}{\sqrt{t}} e^{-a(t+t^{-1})}dt = 2\int_0^\infty e^{-a(t+t^{-1})}d\sqrt{t} = 2\int_\infty^0 e^{-a(t+t^{-1})}d\frac{1}{\sqrt{t}} = \int_0^\infty e^{-a(t+1/t)}d(t^{1/2} - t^{-1/2}) = e^2\int_0^\infty e^{-a(t^{1/2}-t^{-1/2})^2}d(t^{1/2} - t^{-1/2}) = e^2\sqrt{a}\int_{-\infty}^\infty e^{-u^2}du = e^2\sqrt{\pi a}$ where $a = 1985$.

B6. Since for $A = [a_{ij}]$, $\operatorname{tr} A^*A = \sum_{ij} a_{ij}^2$, $\operatorname{tr} A^*A = 0$ implies A. Taking the average of an inner product in $\mathbb{R}^n$ over transformations from G, we obtain a G-invariant inner product, and hence may assume WLOG that matrices M_i are orthogonal: $M_i^* = M_i^{-1}$. Then for each i, $M_i^*M_1, \ldots, M_i^*M_r$ form a permutation of $M_1, \ldots, M_r$. Therefore $(\sum M_i)^*(\sum M_j) = \sum_i M_i^* \sum_j M_j) = r\sum M_j$ and hence has zero trace.