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Math H185. Final Exam. 05.15.14. Solutions

Problem 1. (a) Give integral formuas for coefficients a_n of the Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ of a function f holomorphic in the annulus $R_1 < |z| < R_2$.

Solution. For a fixed z with $R_1 < r_1 < |z| < r_2 < R_2$, the integral of the meromorphic form $\frac{f(t)dt}{t-z}$ over the boundary of the annulus $r_1 \leq |t| \leq r_2$,

$$\oint_{|t|=r_2} \frac{f(t)dt}{t-z} - \oint_{|t|=r_1} \frac{f(t)dt}{t-z},$$

is equal to $2\pi i f(z)$ (by the Residue Theorem). Expanding into geometric series $(t-z)^{-1} = \sum_{n\geq 0} z^n/t^{n+1}$ for $|z| < r_2$ and $-(t-z)^{-1} = \sum_{n<0} z^n/t^{n+1}$ for $|z| > r_1$, and noting that the integrals of holomorphic forms over circles $|t| = r$ don't depend on the radius r, we find:

$$a_n = \frac{1}{2\pi i} \oint_{|t|=r} \frac{f(t)dt}{t^{n+1}}, \quad R_1 < r < R_2.$$

(b) Prove the *Removable Singularity Theorem*: A bounded function holomorphic in the punctured disk $0 < |z| < R$ extends holomorphically to the center of the disk.

Solution. The above formula implies $2\pi |a_n| \leq M/r^n$ for any $R_1 < r < R_2$, where M is an upper bound for $|f$. when $R_1 = 0$, and $n < 0$, this implies (by passing to the limit $r \to 0$) that $a_n = 0$. The inequalities for $n \geq 0$ show that for $|z| < r < R_2$ the power series $\sum_{n\geq 0} a_n z^n$ converges absolutely and uniformly to a holomorphic function which therefore extends f to $z = 0$.

Problem 2. (a) Expand *Koebe's function* $f(z) = z/(1-z)^2$ into a Laurent series in the annulus $|z| > 1$ and in the disk $|z| < 1$.

Solution. For $|z| < 1$, we have: $(1-z)^{-1} = \sum_{n\geq 0} z^n$. Koebe's function is $zd/dz(1-z)^{-1}$. Hence for $|z| < 1$

$$f(z) := \frac{z}{(1-z)^2} = \sum_{n>0} nz^n.$$

Besides, we have: $f(1/w) = f(w)$, and hence for $|z| > 1$

$$\frac{z}{(1-z)^2} = f(1/z) = \sum_{n>0} nz^{-n}.$$

(b) At $z = \infty$, does this function have: a pole, essential singularity, or removable singularity (i.e. extends without singularity)? Why?

Solution. It is a removable singularity (in fact a simple zero). It is clear from the series expansion for $|z| > 1$, or alternatively, since f has a zero at $z = 0$, from the identity $f(1/z) = f(z)$.

Remark. One can easily check that on the unit disk, Koebe's function is simple (= injective). Namely, $f(z_1) = f(z_2)$ implies that either $z_1 = z_2$ or $z_1 = 1/z_2$, that is, if $|z_1| < 1$, then $|z_2| > 1$. In fact Koebe's funtion is extremal among holomorphic functions simple on the unit disk: According to *Bieberbach's conjecture* (1916, proved in 1985 by de Brange), if $g(z) :== z + a_2z^2 + a_3z^3 + ...$ is simple on $z| < 1$, then necessarily $a_n \leq n$.

Problem 3. Let P and Q be two monic polynomials, such that

$$\deg Q = 1 + \deg P.$$

Compute the residue sum:

$$\sum_{z:Q(z)=0} \operatorname{Res}_z \frac{P}{Q} dz.$$

Solution. By the Residue Theorem, the sum of the residues is equal to minus the residue at infinity. Replacing z with $1/w$ in $P = z^n + ...$, $Q = z^{n+1} + ...$, we find:

$$\frac{P(z)}{Q(z)} dz = \frac{w^{-n}(1 + ...)}{w^{-n-1}(1 + ...)} \frac{-dw}{w^2} = -\frac{1 + ...}{1 + ...} \frac{dw}{w},$$

where the dots denote positive powers of w. Thus, the differential form has at $z = \infty$ a simple pole with residue -1. We conclude that the total sum of finite residues is equal to 1.

Problem 4. Compute integral

$$\oint_C \tan(z) dz$$

over cicrle C of radius 4, centered at $z = 3$, and oriented counter-clockwise.

Solution. $\tan z = \sin z / \cos z = -d \log \cos z$. Thus the integral equals $-2\pi i$ times the total number of zeroes of $\cos z$ inside the circle (taking into account that $\cos z$ is entire and hence has no poles). The zeroes are $z = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, ...$ (where $\pi/2 \approx 1.57$). Out of those, only $\pi/2$ and $3\pi/2$ fit in the disk. Thus the integral is equal to $-2 \cdot (2\pi i) = -4\pi i$.

Problem 5. (a) Find an isomorphism of the the domain $G = \mathbf{C} - \mathbf{R}_-$ of the complex plane (consisting of all complex numbers except non-positive real ones) onto the unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$, and transforming $1 \in G$ into $0 \in D$.

Solution. Take the branch of the function $w = \sqrt{z}$ which transforms 1 to 1 and defined on the region G. It transforms G to the right half-plane. Let us compose $w = \sqrt{z}$ with a fractional linear transformation mapping 1 to 0 and the right half-plane to the unit disk D. We may also assume that $w = 0$ is mapped to -1 on the boundary of D. Then the map must have the form $(w-1)/(cw+1)$. When $w = \pm i$, taking $c = 1$ we find the image $(\pm i - 1)/(\pm i + 1)$ to have absolute value 1. Thus, 3 boundary points of the right half-plane are mapped to 3 boundary points of D, and one interior point to one interior. This guarantees that the fractional linear map is an isomorphism of the right half-plane onto D.

(b) Describe *all* isomorphisms $G \to D$ transforming $1 \in G$ into $0 \in D$. Justify your answer.

Solution. Since automorphisms of the unit disk D preserving the center are only the rotations (i.e. multiplications by $e^{i\theta}$, all isomorphisms of G onto D mapping 1 to 0 are given by the formula

$$z \mapsto e^{i\theta}\frac{\sqrt{z}-1}{\sqrt{z}+1}.$$

Problem 6. *The Bernoulli numbers* B_l are defined by the power series expansion:

$$\frac{z}{1-e^{-z}} =: \sum_{l=0}^{\infty} B_l \frac{z^l}{l!}.$$

(a) Find the radius of convergence of the series.

Solution. The ratio has poles at $z = 2\pi i k, k \neq 0$. Hence the convergence radius (i.e. the distance to the closest pole) is 2π.

(b) Show that the only non-zero B_l with odd l is $B_1 = 1/2$, and compute B_0, B_2 and B_4.

Hint: Show that $\frac{z}{1-e^{-z}} - \frac{z}{2} = \frac{z/2}{\tanh(z/2)}$.

Solution. Subtracting $z/2$, we get an even function:

$$\frac{z}{1-e^{-z}} - \frac{z}{2} = \frac{z}{2}\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{z}{2}\frac{\cosh z/2}{\sinh z/2}.$$

We have:

$$\cosh z/2 = 1 + \frac{z^2}{4\cdot 2!} + \frac{z^4}{16\cdot 4!} + ..., \quad \frac{2}{z}\sinh z/2 = 1 + \frac{z^2}{4\cdot 3!} + \frac{z^4}{16\cdot 5!}.$$

Thus

$$\begin{aligned} \frac{1}{1+\frac{z^2}{12\cdot 2!}+\frac{z^4}{80\cdot 4!}+\cdot} &= 1 - \frac{z^2}{12\cdot 2!} + \frac{z^4}{24\cdot 4!} - \frac{z^4}{80\cdot 4!} + \cdots \\ &= 1 - \frac{z^2}{24\cdot 2!} + \frac{7z^4}{240\cdot 4!} + \cdots . \end{aligned}$$

We find:

$$\begin{aligned} \frac{z/2}{\tanh z/2} &= (1 + \frac{z^2}{4\cdot 2!} + \frac{z^4}{16\cdot 4!} + \cdots)(1 - \frac{z^2}{12\cdot 2!} + \frac{7z^4}{240\cdot 4!} + \cdots) \\ &= 1 + \frac{z^2}{6\cdot 2!} - \frac{z^4}{30\cdot 4!} + \cdots , \end{aligned}$$

where we used $1/8 - 1/16 - 7/240 = 8/240 = 1/30$. Thus, $B_0 = 1, B_2 = 1/6$, $B_4 = -1/30$.

(c) Show that the series of meromorphic functions

$$\frac{1}{2} + \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z+2\pi i m} + \frac{1}{z-2\pi i m}\right)$$

converges uniformly on compact subsets to the function $1/(1-e^{-z})$.

Hint: On each strip $|\operatorname{Im} z| \leq M$, using $\sum 1/m^2 < \infty$, prove that the series converges uniformy and tends to $1/2$ as $|\operatorname{Re} z| \to \infty$; then show that the sum is $2\pi i$-periodic, has simple poles at $z = 2\pi i m$ with residue 1, and is completely charactersized by theses properties; finally, verify the same properties for $1/(1-e^{-z})$.

Solution. In fact, since $(1-e^{-z})^{-1} - 1/2 = 1/\tanh z/2$, this problem practically coincides with Example 3 on pages 152-153 of the book, where $1/tan\pi w$ is represented as the sum of a series of meromorphic functions. More precisely, $\tanh z/2 = i \tan \pi w$ where $z = 2\pi i w$. So, I won't copy here the details from the book.

(d) Derive that for $k = 1, 2, 3, ...$

$$\frac{B_{2k}}{(2k)!} = \frac{2\zeta(2k)}{(2\pi)^{2k}}, \text{ where } \zeta(2k) = \sum_{m=1}^{\infty} \frac{1}{m^{2k}},$$

and compute $\sum 1/m^2$ ann $\sum 1/m^4$.

Solution. Multiplying by z, we obtain a series of meromorphic functions converging uniformly on compact susets to the function $z/(1-e^{-z})$ holomorphic at $z = 0$. Expanding each term of the series as a geometric series in $-z^2$ (in the disk $|z^2| < (2\pi)^2$), and comparing the coefficients at z^{2k}, we find:

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \sum_{m \geq 1} \frac{1}{m^{2k}}.$$

In particular, for $k = 1$

$$\frac{1}{12} = \frac{\zeta(2)}{2\pi^2}, \text{ and so } \zeta(2) = \frac{\pi^2}{6},$$

and for $k = 2$

$$-\frac{1}{30 \cdot 4!} = -\frac{\zeta(4)}{8\pi^4}, \text{ and so } \zeta(4) = \frac{\pi^4}{90}.$$