Math 185. Final Exam. 05.08.17

1. Formulate and prove Liouville's theorem.

`Solution 1.` Liouville's theorem says that every bounded function holomorphic on the entire plane $\mathbf{C}$ is constant. To prove it, note that if $|f(t)| < M$ for all $t \in \mathbf{C}$, then the Cauchy integral over circle $C_r(z)$ of radius r centered at z

$$f'(z) = \frac{1}{2\pi i} \oint_{C_r(z)} \frac{f(t)\, dt}{(t-z)^2}$$

in the absolute value does not exceed $2pirM/2\pi r^2 = M/r$ which tends to 0 as $r \to \infty$. Therefore $f'(z) = 0$ for all $z \in \mathbf{C}$, and hence f is constant.

`Solution 2.` A holomorphic function $f : \mathbf{C} \to \mathbf{C}$ expands into a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(t)dt}{t^{n+1}}$$

where the integral is taken (counter-clockwise) over the circle of radius r, which can be any positive number. Assuming that $|f| < M$ everywhere, we find that $|a_n| < M/r^n$. For $n > 1$, passing to the limit $r \to \infty$, we conclude that $a_n = 0$ for all $n > 0$, i.e. that f is constant.

2. Compute

$$\oint_{|z|=2} \frac{z^4\, dz}{z^3 + z + 1},$$

assuming the counter-clockwise orientation of the circle.

`Solution.` For $|z| \geq 2$, we have

$$|z^3 + z + 1| \geq |z|^3 (1 - |z|^{-2} - |z|^{-3}| \geq |z|^3 (1 - 1/4 - 1/8) > 0,$$

i.e. all the 3 finite poles of the integrand are inside the circle $|z| = 2$. Rewriting the integral in terms of $w = 1/z$, we find

$$\oint_{|z|=2} \frac{z^4\, dz}{z^3 + z + 1} = -\oint_{|w|=\frac{1}{2}} \frac{dw}{(1 + w^2 + w^3)w^3} = -\oint_{|w|=\frac{1}{2}} \frac{(1 - w^2 + \cdots)\, dw}{w^3},$$

which is equal to $-2\pi i$, i.e. the *negative* of the residue at the 3-rd order pole $w = 0$, since the circle $|w| = 1/2$ on the w-plane is oriented *clockwise*.

3. Expand $1/(z^2+2)(z^2+4)$ into a power series centered at $z=0$, and find the radius of convergence of this series.

`Solution.`

$$\frac{1}{(z^2+2)(z^2+4)} = \frac{1}{2(z^2+2)} - \frac{1}{2(z^2+4)} = \sum_{n=0}^{\infty}(-1)^n z^{2n}\left(\frac{1}{2^{n+2}} - \frac{1}{2\cdot 4^{n+1}}\right)$$

i.e. the coefficient at z^{2n} is $(-1)^n(1-1/2^{n+1})/2^{n+2}$. The convergence radius can be found from this by the root test, but is obviously equal to $\sqrt{2}$ anyway, since this is the distance from the origin $z=0$ to the closest of the poles $z=\pm i\sqrt{2}, \pm i\sqrt{4}$.

4. Compute

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^6}.$$

`Solution.` Let Γ_r be the closed contour consisting of the interval $[-r,r]$ of the real axis, and the semicircle of radius r centered at the origin and lying in the upper half-plane. For $r>1$, we have

$$\oint_{\Gamma_r}\frac{dz}{1+z^6} = 2\pi i \sum_{\operatorname{Im} z>0, z^6=-1} \operatorname{Res}_z \frac{dz}{1+z^6}.$$

The poles are of the 1st order, so each residue has the form $1/6z^5 = z/6z^6 = -z/6$. The poles in in the upper half plane are located at $z=i, (\pm\sqrt{3}+i)/2$, and the contour integral equals

$$-\frac{2\pi i}{6}\left(i+\frac{\sqrt{3}}{2}+\frac{i}{2}-\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = \frac{2\pi}{3}.$$

This is actually the value of the integral in question (which is the limit of $\int_{-r}^{r} dx/(1+x^6)$ as $r\to\infty$, because the integral over the semicircle

$$\left|\int_{\theta=0}^{\pi}\frac{dre^{i\theta}}{1+r^6e^{6i\theta}}\right| < \pi r^{-5}$$

tends to 0 as $r\to\infty$.

5. Give the definition of an essential isolated singularity, and prove Casorati-Weierstrass' theorem: If function $w=f(z)$ has an essential isolated singularity at z_0, then the image of every punctured neighborhood of z_0 is dense in $\mathbf{C}$.

`Solution.` A function is said to have an *essential* singularity at $z = z_0$, if it is holomorphic in a puncture neighborhood of z_0, and the function's Laurent series centered at z_0 has infinitely many non-zero coefficients at monomials $(z - z_0)^k$ of negative degree k. Note that if the number of such non-zero coefficients is finite, the function has a pole or removable singularity at z_0, and in either case has a limit as $z \to z_0$, infinite (in the case of the pole) or finite.

Assuming that the claim of Casorati-Weierstrass' theorem is false, we have a punctured neighborhood of z_0 (of some radius $\delta > 0$) whose image does not intersect a neighborhood (of some radius $\epsilon > 0$) of some $w_0 \in \mathbf{C}$. Then $g(z) := 1/(f(z) - w_0)$ is bounded in the absolute value by $1/\epsilon$ in the punctured δ-neighborhood of z_0. This implies that g has a removable singularity at z_0, and so $f(z) = w_0 + 1/g(z)$ has either pole at z_0 or a removable singularity, but not the essential one.

6. Find all conformal automorphisms of the unit disk $|z| < 1$ which map 0 to a given point a inside the disk.

`Solution.` The answer is

$$z \mapsto \frac{ze^{i\theta} + a}{1 + \bar{a}ze^{i\theta}}.$$

Indeed, all automorphisms of the unit disk are fractional linear, and all those which map 0 to a are obtained from one of them (e.g. $(z + a)/(1 + \bar{a}z)$) by precomposing it with those which preserve 0. The latter are rotations $z \mapsto ze^{i\theta}$ (by Schwarz' lemma). To make sure that the fractional linear transformation $w = (z + a)/(1 + \bar{a}z)$ is the disk's automorphism, we note that when $|z| = 1$, we have $z^{-1} = \bar{z}$, and hence $|w|/|z| = |1 + a\bar{z}|/|1 + \bar{a}z| = 1$ as the ratio of complex-conjugate numbers.