Sample Final Exam

Problem 1. (a) Give integral formulas for coefficients a_n of the Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ of a function f holomorphic in the annulus $R_1 < |z| < R_2$.

(b) Prove the *Removable Singularity Theorem*: A bounded function holomorphic in the punctured disk $0 < |z| < R$ extends holomorphically to the center of the disk.

Problem 2. (a) Expand Kobe's function $f(z) = z/(1-z)^2$ into a Laurent series in the annulus $|z| > 1$.

(b) At $z = \infty$, does this fiunction have at infinity: a pole, essential singularity, or removable singularity? Why?

Problem 3. Given a function f holomorphic in the unit disk D, and such that for every point $z \in D$, there exists an $n \in \mathbf{N}$ such that the nth derivative of f vanishes at z. Prove that f is a polynomial.

Problem 4. Let P and Q be two polynomilas, such that

$$\deg Q - \deg P > 1.$$

Prove that the total sum of the residues of the rational function $P(z)/Q(z)$ over the points where the denominator Q vanishes, is equal to zero:

$$\sum_{z:Q(z)=0} \operatorname{Res}_z \frac{P}{Q} = 0.$$

Problem 5. Compute integral

$$\oint_C \tan(z) dz$$

over cicrle C of radius 4, centered at $z = 0$, and oriented counter-clockwise.

Problem 6. (a) Find a conformal equivalence of the the domain $G = \mathbf{C} - \mathbf{R}_-$ of the complex plane (consisting of all complex numbers except non-positive real ones) onto the unit disk $D = \{z \in \mathbf{C} \mid |z| < 1\}$, and transforming $1 \in G$ into $0 \in D$.

(b) Describe *all* conformal equivalences $G \to D$ transforming $1 \in G$ into $0 \in D$. Justify your answer.