Answers to HW5

58. Let $G$ be a $p$-group. The conjugacy class of $g \in G$ (as an orbit of $G$-action) has cardinality divisible by $p$ unless $g$ lies in the center $Z$ of $G$ (i.e. commutes with all elements of $G$). Therefore $|Z|$ is also divisible by $p$. If $g \in G - Z$, then the stabilizer of $g$ under conjugations includes $g$ and $Z$, and hence has order $p^2$ at least. When $|G| = p^2$, this contradicts the assumption that $g$ is not in $Z$, leading to the conclusion that $Z = G$, i.e. $G$ is abelian.

60. $G$ contains 8 rotations through 120° leaving 3^2 fixed colorings each, 6 rotations through 90° leaving 3^3 fixed colorings each, 6 rotations through 180° about the lines passing through midpoints of opposite edges and leaving 3^3 fixed colorings each, 3 rotations through 180° about the axes passing through the centers of opposite faces and leaving 3^4 fixed colorings each, and the identity which leaves all 3^6 colorings fixed. Thus, the number of distinct colorings is

$$\frac{8 \cdot 3^2 + 6 \cdot 3^3 + 6 \cdot 3^3 + 3 \cdot 3^4 + 3^6}{24} = 57.$$ 

On the other hand: one color can be applied to all faces (3 colorings totally). Using 2 colors: the first one applied to one face (6), to two opposite (6) or two adjacent faces (6), to three faces adjacent to one vertex (3) or not (3) which gives totally 24 colorings. Three colors can be applied to 1 + 1 + 4 faces respectively, the first two to opposite or adjacent ones ($3 + 3 = 6$ colorings). Or, they can be applied to 3 + 2 + 1 faces respectively: the first one near one vertex (6) — or not, in which case the second one can be applied to opposite (6) or adjacent faces (6) yielding totally 18 colorings. Alternatively, three colors can be applied to 2 + 2 + 2 faces — either to pairs of opposite faces (1) or one to opposite the other two to pairs of adjacent faces (3), or to a pair of adjacent faces each (1), i.e. totally 5. Combining we find: $3 + 24 + 6 + 18 + 5 = 56$. Oops! In fact a configuration of 3 distinct colors, applied to 2 pairs of adjacent faces each, defines an orientation of space and hence yields two types of colorings not transformable into each other by the cube’s rotations. This explains the missing 57th item.

62. In $A_4$, there are 8 3-cycles which generate 4 cyclic Sylow 3-subgroups. The remaining 4 elements belong to the normal Klein subgroup $K_4 \subset A_4$, which is therefore the Sylow 2-subgroup.

In $D_6$, rotations through $\pm 180^\circ$ generate the cyclic Sylow 3-subgroup. There are no elements of order 4 in $D_6$. A pair of reflections about perpendicular lines (one passing through a pair of opposite vertices of the hexagon, the other through midpoints of opposite edges) generate one
of 3 Sylow 2-subgroups isomorphic to $K_4$. (Note that 3 is the largest odd divisor of 12.)

65. A Sylow $p$-subgroup $S_p$ is unique, since none of the non-trivial divisors of $pq$ is congruent to 1 mod $p$. The same applies to Sylow $q$-subgroup $S_q$ (since $1 < p < q$). Thus, both are normal. Then take $g = ab$ where $a$ and $b$ are non0-identity elements in $H_p$ and $H_q$ respectively. In projection to $G/H_p$, $g$ has order $q$ and in projection to $G/H_q$, $g$ has order $p$. Therefore the order of $g$ is $pq$, and hence $G$ is cyclic.