Answers to HW1

2. For a transitive and reflexive but not symmetric binary relation, one can take \( \leq \); for transitive and symmetric but not reflexive, take the empty relation on any non-empty set; for reflexive and symmetric but not transitive, take the relation on the set of words of the English language, to have a letter in common.

4. G.C.D.\((1763, 991) = 1 = 181 \times 1763 - 322 \times 991\), and hence \(991^{-1} \equiv -322 \equiv 1441 \mod 1763\).

5. On \(\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}\), the norm \(N(a + b\sqrt{-5}) := (a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2\) takes values \(0, 1, 4, 5, 6, \ldots\), but cannot be equal \(2\) or \(3\). Therefore \(2\) and \(3\), which divide \((1 + \sqrt{-5})(1 - \sqrt{-5}) = 6\) but don’t divide \(1 \pm \sqrt{-5}\), are not prime. However, they are irreducible, since \(N(2) = 4\) and \(N(3) = 9\) cannot be factored into the product of norms \(N(\alpha)N(\beta)\) in any nontrivial way, i.e. with both \(N(\alpha), N(\beta) > 1\).

9. A permutation of \(A, B, C, D\) inducing a trivial permutation of \(V, H, S\) must keep vertical edges vertical, horizontal horizontal, and diagonals diagonal, i.e. must come from a geometric symmetry of the rectangle: reflection about horizontal axis (\(R_h\)), vertical axis (\(R_v\)), their composition (which is the central symmetry, \(C_s\) or equivalently the rotation through \(180^\circ\)), or the identity \(Id\). Composing these four with any particular permutation \(\sigma\) of \(A, B, C, D\) will permute \(H, V, S\) the same way. Thus, all \(4! = 24\) permutations of \(A, B, C, D\) are partitioned into groups of four, where \(\sigma\) and \(\sigma'\) are in the same group whenever \(\sigma^{-1}\sigma' \in \{R_h, R_v, C_s, Id\}\). The last condition is equivalent to saying that \(\sigma\) and \(\sigma'\) induce the same permutation of \(V, H, S\). Since \(24/4 = 3!\), each permutation of \(V, H, S\) is so induced four times.