180. Adjoining to $F := \mathbb{Z}_p(t)$ a root $\theta$ of the polynomial $f(x) := x^p - t \in F[x]$ results in the algebraic extension $F(\theta) \supset F$ which is non-separable. Indeed, $f$ factors over $F(\theta)$ as $x^p - t = (x - \theta)^p$. But $f$ is irreducible over $F$, since $\theta$ (and hence some coefficient of $(x - \theta)^k = x^k - k\theta x^{k-1} + \cdots$ for $0 < k < p$) does not lie in $F$. Indeed, since $t$ is transcendental over $\mathbb{Z}_p$, no rational function $a(t)/b(t)$ satisfies $(a(t)/b(t))^p = t$ (i.e. $a(t^p) = tb(t^p)$).

182. The minimal polynomial over $\mathbb{Q}$ for $\alpha$ is $f = x^3 - 2$, and for $\beta$ is $x^2 - 2$. The sufficient condition for $\theta = \sqrt[4]{2} + \sqrt[4]{2}$ to serve as a primitive element is that it is not equal to any other sum $\alpha' + \beta'$ of roots of the respective polynomials. And indeed, the other root $-\sqrt[4]{2}$ of $g$ is also real, while the other two roots of $x^3 - 2$ are non-real. Thus $a + \beta = \alpha' + \beta'$ is impossible unless $\alpha'$ is real (hence $= \alpha$), in which case $\beta' = \beta$.

Now, $h(x) := g(\theta - x) = (th - x)^2 - 2 = x^2 - 2tx + (\theta^2 - 2)$ has $\alpha = \sqrt[4]{2}$ as a root, and must have a common factor with $f(x) = x^3 - 2$. Performing the Euclidean algorithm, we have:

$$f(x) - (x + 2\theta)h(x) = (3\theta^2 + 2)x + (-2\theta^3 + 4\theta - 2).$$

Since $\alpha$ is root of the L.H.S., we find $\sqrt[4]{2} = (2\theta^3 - 4\theta + 2)/(3\theta^2 + 2)$. This gives an expression of $\sqrt[4]{2}$ as an element of $\mathbb{Q}(\theta)$, and $\sqrt[2]{2} = \theta - \sqrt[4]{2} = (\theta^3 + 6\theta - 2)/(3\theta^2 + 2)$ also lies in $\mathbb{Q}(\theta)$.

189. (a) The field $E$ can be described as $\mathbb{Q}(\sqrt[4]{2}, i)$. It contains the subfield $\mathbb{Q}(\sqrt[4]{2}) \cong \mathbb{Q}[x]/(x^4 - 2)$ of degree 4 over $\mathbb{Q}$ (e.g. because the polynomial $x^4 - 2$ is irreducible by Eisenstein’s criterion with $p = 2$). Since $\mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$ contains $\pm \sqrt[4]{2}$ but does not yet contain the other two roots $\pm i\sqrt[4]{2}$ of $x^4 - 2$, the splitting field must be obtained by adjoining $i = \sqrt{-1}$ (the ratios of the roots), and thus coincides with $\mathbb{Q}(\sqrt[4]{2}, i)$, which has therefore degree 8 over $\mathbb{Q}$. For a basis, putting $\alpha := \sqrt[4]{2} \in \mathbb{R}_{>0}$, one can take $1, \alpha, \alpha^2, \alpha^3, i, i\alpha, i\alpha^2, i\alpha^3$.

(b) In fact it is clear a priori that the Galois group $G(E/\mathbb{Q})$, being a subgroup of order 8 in the group $S_4$, must be non-abelian, isomorphic to the dihedral group $D_4$. Indeed, since $|S_4| = 24$, a subgroup of order 8 must be one of the Sylow 2-subgroups, which are all conjugated (hence isomorphic), and in the rotation group of the cube (isomorphic to $S_4$), there are three such dihedral groups (formed by rotations of the cube interpreted as a square prism).
To describe the action of $D_4$ on the roots of $x^4 - 2$, consider the roots $\alpha, i\alpha, -\alpha, -i\alpha$ (we remind that $\alpha = \sqrt[4]{2}$) as the vertices of a square on the complex plane. The automorphism of $E$ induced by mapping $i$ to $i$ and $\sqrt[4]{2}$ to $i\sqrt[4]{2}$ induces the counter-clockwise rotation of this square through $90^\circ$. The complex conjugation $i \mapsto -i$ (and $\sqrt[4]{2} \mapsto -\sqrt[4]{2}$) induces the reflection of the square about the real axis. The two transformations generate the dihedral (Galois) group.

(c) The groups $D_4$ has 3 subgroups of index 2 (all therefore normal): one isomorphic to $\mathbb{Z}_4$ and consisting of the rotations of the square, and 2 isomorphic to $\mathbb{Z}_2^2$, each consisting of two conjugated reflections of the central symmetry ($= \text{rotation through } 180^\circ$). The latter rotation also generates the normal subgroup $\mathbb{Z}_2$ (which happens to form the center of $D_4$), and each of the 4 reflections generate 4 non-normal subgroups isomorphic to $\mathbb{Z}_2$. The list of totally 10 subgroups is completed by the trivial group $\{e\}$.

The last subgroup corresponds of course to the whole field $E^{(e)} = E$. When the subgroup $H$ is the normal $\mathbb{Z}_2$ (generated by the central symmetry) of the rectangle), that symmetry maps $i \mapsto i$, and $\alpha := \sqrt[4]{2} \mapsto -\alpha = -\sqrt[4]{2}$. In our basis $1, \alpha, \alpha^2, i\alpha, i\alpha^2, i\alpha^3$, this symmetry acts by preserving $1, i, \alpha^2, i\alpha^2$ and changing the sign of $\alpha, \alpha^3, i\alpha, i\alpha^3$. The fixed points of it form the subspace spanned by $1, i, \alpha^2, i\alpha^2$, i.e. the subfield $\mathbb{Q}(\sqrt[4]{2}, i)$ of degree 4 over $\mathbb{Q}$.

A similar analysis of conjugated reflections $\alpha, i\alpha \mapsto \alpha, -i\alpha$ and $\alpha, i\alpha \mapsto -\alpha, i\alpha$ shows that the fixed points form two conjugate subfields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(i\alpha)$ of degree 4 over $\mathbb{Q}$.

Analyzing fixed points of the other two conjugated reflections, which are $\alpha, i\alpha \mapsto i\alpha, \alpha$ and $\alpha, i\alpha \mapsto -i\alpha, -\alpha$ (both mapping $i \mapsto -i$), we will conclude that the corresponding fields are isomorphic to $\mathbb{Q}[x]/(x^4 + 2)$ and coincide with $\mathbb{Q}(\alpha(1 - i)/\sqrt[4]{2})$ and $\mathbb{Q}(\alpha(1 + i)/\sqrt[4]{2})$ respectively. (Namely, the 1st reflection interchanges $\alpha^3$ with $-i\alpha^3$, and hence preserves their average $\alpha(1 - i)/\sqrt[4]{2}$, while the 2nd likewise preserves $\alpha(1 + i)/\sqrt[4]{2}$.)

The intersection of each of these 4 degree-4 subfields of $E$ with the normal subfield $\mathbb{Q}(\sqrt[4]{2}, i)$ yields the 2 degree-2 extensions of $\mathbb{Q}$ fixed by the 2 subgroups of $D_4$ isomorphic to $\mathbb{Z}_2^2$: $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$.

The subfield $\mathbb{Q}(i)$ corresponds to $\mathbb{Z}_4 \subset D_4$ (mapping $\alpha \mapsto i\alpha \mapsto -\alpha \mapsto -i\alpha \mapsto \alpha$, but fixing $i$), and of course the whole $D_4$ corresponds to $E^{D_4} = \mathbb{Q}$. 


190. \( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \) is the splitting field of the collection \( x^2 - 2, x^2 - 3, x^2 - 5 \) of three quadratic polynomials; so it is normal, is obtained by consecutively adjoining to \( \mathbb{Q} \) the three square roots, has the basis \( 1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{2} \cdot 3, \sqrt{2} \cdot 5, \sqrt{3} \cdot 5, \sqrt{2} \cdot 3 \cdot 5 \) over \( \mathbb{Q} \), and Galois group isomorphic to \( \mathbb{Z}_2^3 \) and generated by independent changes of the signs of \( \sqrt{2}, \sqrt{3}, \sqrt{5} \mapsto \epsilon_1 \sqrt{2}, \epsilon_2 \sqrt{3}, \epsilon_3 \sqrt{5} \), where \( \epsilon_i = \pm 1 \).

The group \( \mathbb{Z}_2^3 \) can be considered as a 3-dimensional vector space over \( \mathbb{Z}_2 \), and its subgroups as \( \mathbb{Z}_2 \)-subspaces: there are 7 subspaces of dimension 1 (spanned by 7 non-zero vectors), and 7 subspaces of dimension 2 (given by 7 non-zero linear equations).

The 7 subfields, whose elements are fixed a corresponding non-trivial element \( \epsilon = (\pm 1, \pm 1, \pm 1) \neq (1, 1, 1) \) of the Galois group are:

\[
\begin{align*}
(−1, 1, 1) &: \mathbb{Q}(\sqrt{3}, \sqrt{5}), & (1, −1, 1) &: \mathbb{Q}(\sqrt{2}, \sqrt{5}), \\
(1, 1, −1) &: \mathbb{Q}(\sqrt{2}, \sqrt{3}), & (−1, −1, 1) &: \mathbb{Q}(\sqrt{3}, \sqrt{6}), \\
(−1, 1, −1) &: \mathbb{Q}(\sqrt{3}, \sqrt{10}), & (1, −1, −1) &: \mathbb{Q}(\sqrt{3}, \sqrt{10}), \\
(−1, −1, −1) &: \mathbb{Q}(\sqrt{6}, \sqrt{10}) & \text{ (note that } \sqrt{15} = \sqrt{6}\sqrt{10}/2). \\
\end{align*}
\]

Writing \( \epsilon = ((−1)^a, (−1)^b, (−1)^c) \) with \( a, b, c \equiv 0, 1 \mod 2 \), we have 7 non-trivial linear equations of 2-dimensional subgroups, and the corresponding subfields of their fixed elements:

\[
\begin{align*}
a &\equiv 0 : \mathbb{Q}(\sqrt{2}), & b &\equiv 0 : \mathbb{Q}(\sqrt{3}), & c &\equiv 0 : \mathbb{Q}(\sqrt{5}), \\
a + b &\equiv 0 : \mathbb{Q}(\sqrt{6}), & a + c &\equiv 0 : \mathbb{Q}(\sqrt{10}), & b + c &\equiv 0 : \mathbb{Q}(\sqrt{15}), \\
a + b + c &\equiv 0 : \mathbb{Q}(\sqrt{30})
\end{align*}
\]