George M. Bergman Supplementary material

Some notes on sets, logic, and mathematical language

These are "generic" notes, for use in Math 110, 113, 104 or 185.

These pages do not develop in detail the definitions and concepts to be mentioned. That is done, to various degrees, in Math 55, Math 74, Math 125 and Math 135. I hope you will nevertheless find these notes useful and thought-provoking. I recommend working the exercises for practice; but don't hand them in unless they are listed in a homework assignment for the course.

1. Set-theoretic symbols

Symbol.	*Meaning, usage, examples, discussion.*
$\mathbf{N}$, $\mathbf{Z}$	Here $\mathbf{N}$ denotes the set of all *natural numbers*, i.e., $\{0, 1, 2, 3, ...\}$, while $\mathbf{Z}$ (from "Zahl", German for "number") denotes the set of all *integers,* $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. (Many older authors started the natural numbers with 1, but it is preferable to start with 0, since natural numbers are used to count the elements of finite sets, and the set with no elements is a finite set.)
$\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$	Of these, $\mathbf{Q}$ (for "quotient") denotes the set of all *rational numbers*, i.e., fractions that can be written with integer numerator and denominator, $\mathbf{R}$ denotes the set of *real numbers*, and $\mathbf{C}$ the set of *complex numbers*. (The above five boldface letters are generally shown on the blackboard by writing the given letter with one main stroke doubled: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Since it is convenient to have distinctive symbols for these five important sets, printed forms imitating these symbols have been created, generally called "blackboard bold", which are now used in many works. However, in handouts in this course, I will follow our text and use regular boldface symbols $\mathbf{Z}$, $\mathbf{Q}$ etc.)
$\in$	"Is a member of". E.g., $3 \in \mathbf{Z}$.
$\{\ \}$	"The set of all". This is often used together with " $\mid$ " or ":" (different authors use one or the other), which stand for "such that". For instance, the set of *positive* integers can be written $\{1, 2, 3, ...\}$ or $\{n \in \mathbf{Z} \mid n > 0\}$ or $\{n : n \in \mathbf{Z}, n > 0\}$. The set of all square integers can be written $\{0, 1, 4, 9, ..., n^2, ...\}$ or $\{n^2 \mid n \in \mathbf{Z}\}$. Note also that $\{n^2 \mid n \in \mathbf{Z}\} = \{m^2 \mid m \in \mathbf{Z}\} = \{m^2 + 2m + 1 \mid m \in \mathbf{Z}\}$. (Why?)
$\varnothing$	The empty set; i.e., the set which contains no elements.
$\subseteq$	"Is a subset of". E.g., $\varnothing \subseteq \{1\} \subseteq \mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{R} \subseteq \mathbf{C}$. $\{n^2 \mid n \in \mathbf{Z}\} \subseteq \{n \in \mathbf{Z} \mid n \geq 0\}$. $\mathbf{Z} \subseteq \mathbf{Z}$.
$\subset$ or $\subsetneqq$	"Is a proper subset of"; that is, a subset that is not the whole set. For instance, $\mathbf{Z} \subsetneqq \mathbf{R}$. In fact, all the formulas used above to illustrate "$\subseteq$" remain true with $\subsetneqq$ in place of $\subseteq$ except for $\mathbf{Z} \subseteq \mathbf{Z}$. Since a proper subset is, in particular, a subset, one may use $\subseteq$ even when $\subsetneqq$ is true; and one generally does so, unless one wants to emphasize that a subset is proper. But beware: some authors (especially in Eastern Europe) use $\subset$ for "is a subset of".
$\supseteq$, $\notin$, etc.	Obvious variants of the above symbols: $A \supseteq B$ means $B \subseteq A$; $x \notin X$ means x is not a member of X; $A \nsubseteq B$ means A is not a subset of B; etc.. Warning: The phrases "A lies in X" and "A is contained in X" can each mean either $A \in X$ or $A \subseteq X$. (The former phrase more often means $A \in X$ and the latter $A \subseteq X$, but this is no guarantee.) So in your writing, if there is danger of ambiguity, either use the symbol, or use the unambiguous phrase "is a member of" or "is a subset of".
$\cap$	Intersection: $A \cap B = \{x \mid \text{both } x \in A \text{ and } x \in B \text{ are true}\}$. For instance, $\{x \in \mathbf{Z} \mid x > 9\} \cap \{x \in \mathbf{R} \mid x \leq 12\} = \{10, 11, 12\}$.
$\bigcap$	Intersection of an indexed family of sets. For instance, if $A_0, A_1, ...$ are sets, then $\bigcap_{n=0,1,...} A_n$, also written $\bigcap_{n \in \mathbf{N}} A_n$, $\bigcap_{n=0}^{\infty} A_n$ and $A_0 \cap A_1 \cap ... \cap A_n \cap ...$, means $\{x \mid x \text{ is a member of all of } A_0, A_1, \text{etc.}\}$.

In an intersection $\bigcap_{i\in I} A_i$, I does not have to be a set of integers; it can be any set such that A_i is defined for each $i \in I$. When a set I is used in this way to index (i.e., list) other entities, it is called an *index set*.

$\cup$ Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. For instance, $\{x \in \mathbf{Z} \mid x > 0\} \cup \{x \in \mathbf{Z} \mid x < 12\} = \mathbf{Z}$. Note that if $A \subseteq B$ then $A \cup B = B$ and $A \cap B = A$.

$\bigcup$ Union of an indexed family of sets. Thus $\bigcup_{n=0,1,\ldots} A_n = \bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup \ldots \cup A_n \cup \ldots$. Example: $\bigcup_{n\in\mathbf{N}} \{i \in \mathbf{N} \mid i < n\} = \mathbf{N}$.

Often, when the intent is clear from context, the above notations are abbreviated. For instance, if we know that we have one set A_i for each i in a certain index-set I, then instead of $\bigcup_{i\in I} A_i$, we may write $\bigcup_I A_i$ or $\bigcup_i A_i$ or just $\bigcup A_i$. Likewise, if we have a set X_n for each positive integer n, the intersection $\bigcap_{n=1}^{\infty} X_n$ may be written $\bigcap_n X_n$ or simply $\bigcap X_n$.

^{c} "Complement of": ${}^{c}A$ means $\{x \mid x \notin A\}$.

But $\{x \mid x \notin A\}$ makes no sense if we don't say what x's are allowed! So the notation ${}^{c}A$ is used only when discussing subsets of a fixed set. For instance, if we are discussing subsets of the integers, then ${}^{c}\{\text{even integers}\} = \{\text{odd integers}\}$, while if we are considering subsets of $\mathbf{Q}$, then ${}^{c}\{\text{even integers}\}$ denotes the set consisting of all odd integers and non-integer rationals. To be more precise, we can use the next symbols:

$-$ or $\backslash$ $A - B$ (or $A \backslash B$) means $\{x \in A \mid x \notin B\}$. E.g., $\mathbf{Z} - \{\text{even integers}\} = \{\text{odd integers}\}$. Note that $A - B$ is defined even if B is not a subset of A. For instance, $\mathbf{Z} - \{\text{negative real numbers}\} = \mathbf{N}$.

$f\colon X \to Y$ This indicates that f is a function (also called a 'map' or 'mapping') from the set X to the set Y. (In reading the symbol out loud, one can use words such as "the map f from X to Y", or "f sending X to Y".)

Such an f is said to be *one-to-one* (or *injective*) if for every two distinct elements $x_1, x_2 \in X$, the elements $f(x_1)$ and $f(x_2)$ of Y are also distinct. For instance, the operation of cubing an integer is a one-to-one function $\mathbf{Z} \to \mathbf{Z}$; but the squaring map is *not* one-to-one, because $(-n)^2 = n^2$.

The function $f\colon X \to Y$ is said to be *onto* Y if every element of Y equals $f(x)$ for some $x \in X$. For instance, the squaring and cubing maps $\mathbf{Z} \to \mathbf{Z}$ are not onto, since not all integers are squares or cubes. On the other hand, the cubing map $\mathbf{R} \to \mathbf{R}$ is both one-to-one and onto.

Given $f\colon X \to Y$, the set X is called the *domain* of f. What about the set at the other end of the arrow? A complication is that if f is not onto Y, then Y and $\{f(x) \mid x \in X\}$ are different sets. Traditionally, these were called the "range" and the "image" of f respectively, but the usage was not firm; "range" was often used as a synonym for "image". Hence nowadays, the unambiguous term "codomain" has been introduced to describe Y. A function is called *onto* if it is *onto* its codomain; a synonymous term is "surjective" (from French "sur" = "onto").

A map $f\colon X \to Y$ that is both one-to-one and onto has an inverse map $g\colon Y \to X$, taking each $y \in Y$ to the unique element $x \in X$ such that $f(x) = y$. Thus, f may be thought of as defining a matching $X \longleftrightarrow Y$, under which each element of X is matched with a unique element of Y and vice versa; such a matching is called a *one-to-one correspondence* between X and Y. So the phrases "one-to-one and onto function", "invertible function" and "one-to-one correspondence" all describe the same thing; still another term for this is "*bijective* map" ("bijective" meaning "both injective and surjective").

f^{-1} This symbol is used in *three different ways*, which are related, but not quite the same. A lot of confusion can result if they are not distinguished. First, if f is one-to-one and onto, then f^{-1} denotes the *inverse* of f, discussed above. Secondly, if $f\colon X \to Y$ is any map, and S is any subset of Y, then $f^{-1}(S)$ denotes $\{x \in X \mid f(x) \in S\}$, called the *inverse image* (or *preimage*) of S under f. When f is invertible it is not hard to check that this is precisely the image of the set S under the inverse function $f^{-1}: Y \to X$. However, this definition of $f^{-1}(S)$ makes sense even when f is not invertible. Finally, for $y \in Y$, the symbol $f^{-1}(\{y\})$ is often abbreviated to $f^{-1}(y)$. Hence when the symbol "f^{-1}" is used, you must check whether the context indicates that f is an invertible function. If so, you can be confident that f^{-1} denotes the inverse function; if not, then f^{-1} does not stand for a function, but is a way of writing inverse images of sets or elements under f.

(Note also that the symbol $f(x)^{-1}$, which can turn up when f is, say, a real-valued function, means something unrelated to the above three concepts: it denotes the multiplicative inverse of the real number $f(x)$. If, in such a situation, you want to refer to the function that takes x to $f(x)^{-1}$, you can't call it f^{-1}; but you could write it $1/f$.)

$\mapsto$ While the ordinary arrow referred to above is used to show what the domain and codomain of a function are, the "flat-tailed" arrow shows which element is carried to which. Thus, $f: x \mapsto x^2$ means that f is the squaring function, defined by $f(x) = x^2$. We can use this kind of arrow to describe a function without denoting it by a letter, e.g., "the function $x \mapsto x^2$" (which can be read "the function x-goes-to-x^2" or "the function taking x to x^2").

$\times$ If A and B are sets, $A \times B$ means the set of ordered pairs $\{(a, b) \mid a \in A,\ b \in B\}$. I won't discuss here precisely how an ordered pair is defined; simply think of it as a "list" of length 2. Note that a function f of two variables, one A-valued and one B-valued, which takes values in a set C, can be thought of as a map $f: A \times B \to C$. For example, addition of integers is a map $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, given by $(m, n) \mapsto m + n$, while exponentiation of real numbers using natural numbers as exponents, given by $(x, n) \mapsto x^n$, is a map $\mathbf{R} \times \mathbf{N} \to \mathbf{R}$.

Likewise, $A \times B \times C$ denotes the set of ordered 3-tuples $\{(a, b, c) \mid a \in A,\ b \in B,\ c \in C\}$, and one defines a function of three variables as a map on such a triple product-set; and so on.

The set $A \times B$ is called the *product* (or *direct product*) of the sets A and B because if A and B are finite, A having m members and B having n members, then $A \times B$ will have mn members. In analytic geometry one regards the set $\mathbf{R} \times \mathbf{R}$ of pairs of real numbers as labeling the points of the plane. The numbers so used are called the "Cartesian coordinates" of the points, after René Des***cartes*** who discovered this approach to geometry. Hence one often calls the direct product $A \times B$ of two sets their *Cartesian product*.

If $f: X \to Y$ is a function, then $\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$, is called the *graph* of f – again, the idea comes from analytic geometry.

Exercise 1. Let X and Y be sets. Find conditions on a subset $S \subseteq X \times Y$ that are necessary and sufficient for S to be the graph of a function $X \to Y$. (I.e., necessary and sufficient conditions on S for there to exist a function $f: X \to Y$ such that S is the graph of f.) If these conditions hold, will S uniquely determine the function?

In view of the answer to the above exercise, set-theorists often *define* a function from a set X to a set Y as a subset of $X \times Y$ having the appropriate properties.

The sets $A \times A$, $A \times A \times A$, etc. are often written A^2, A^3, etc.. Also, if A and B are sets, the set of functions from A to B is often written B^A, since if A is a finite set with m elements and B a finite set with n elements, this set of functions will have n^m elements.

2. Logical connectives

Let us begin by noting that there is a kind of inverse relation between statements and possibilities – the more statements we make, the more we limit the set of possibilities; the more possibilities there are, the more limited is the set of true statements.

For instance, suppose we are considering an integer-valued variable x. If we require that x be positive, then the set of possibilities we are considering forms the set $\{1, 2, 3, \ldots\}$. If instead we had stated that x is even, then the set would have been $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$. If we impose *both* conditions, then the only possibilities for x are the *positive even* integers, a proper subset of each of those sets. More generally, if P and Q are any two conditions, and we assume that P *and* Q hold, then the set of cases we are allowing is the *intersection* of the set of cases allowed by P alone and the set of cases allowed by Q alone. This will help you remember the next symbol, which is similar to $\cap$.

$\wedge$ "And": If P and Q are two statements, then we define "$P \wedge Q$" to hold if and only if P and Q *both* hold. For example, the condition $0 \le x \le 1$ is an abbreviation of $(0 \le x) \wedge (x \le 1)$. The operation $\wedge$ is called "conjunction".

On the other hand, if we want to consider all cases allowed by a condition P and *also* all cases allowed by Q – the union of the two sets of cases – then we are considering the condition "P *or* Q holds". This is a weaker condition than either P or Q, in line with the "inverse relation" noted above between statements and cases. The symbol used is similar to $\cup$, namely

$\vee$ "Or": If P and Q are two statements, we say that "$P \vee Q$" holds in a situation if P holds or if Q holds (possibly both). For instance, for all real numbers x, we have $(x < 10) \vee (x > 0)$. The condition $x \le y$ is equivalent to $(x < y) \vee (x = y)$. As another example, for all integers a, b we have $(a \ge 0) \vee (b \ge 0) \vee (ab \ge 1)$. The operation $\vee$ is called "disjunction".

The "inverse relation" between statements and possibilities that we have mentioned is sometimes a cause of confusion. Many precalculus students, when asked to say what real numbers x satisfy $x^2 > 1$, will describe these as the set of x satisfying "$x > 1$ and $x < -1$". What they mean is that the set *consists of* all x satisfying $x > 1$ *and* all x satisfying $x < -1$. The correct way to express this union of cases is not by the conjunction "$x > 1$ and $x < -1$", but by the disjunction "$x > 1$ or $x < -1$". That is:

$$\{x \mid x^2 > 1\} = (-\infty, -1) \cup (1, +\infty) = \{x \mid x < -1\} \cup \{x \mid x > 1\} = \{x \mid (x < -1) \vee (x > 1)\}.$$

$\neg$ "Not". E.g., $\neg(x = y)$ means $x \ne y$.

$\Rightarrow$ "Implies". For instance, if $x \in \mathbf{R}$, then $x > 2 \Rightarrow x > 0$.

If P and Q are statements, then "$P \Rightarrow Q$" is a statement which is considered to be true in all cases except those where P is true but Q is false. $P \Rightarrow Q$ may also be expressed in words "If P then Q" or "Q if P". For instance, the true statement "$x > 2 \Rightarrow x > 0$" can expressed "$x > 0$ if $x > 2$".

There are certain conventions in the everyday (nonmathematical) use of words such as "or" and "if", which we follow unconsciously, but need to become aware of so as to understand that they do *not* apply to mathematical usage.

In everyday life, we generally use the word "or" only when we do *not know* which of the two possibilities is true. E.g., if a letter comes in the mail and you say, "This is either from John or Stephanie", you are asserting that it is from one of them, but also, implicitly, that you do not yet know *which* one. There are variants on this convention: If you say "I am holding a penny in either my right hand or my left hand", then *you* know which hand it is in, but the person you are speaking to does not. On the other hand, a mathematical statement $P \vee Q$ is considered to be true even when we *do* know which of P or Q holds. For instance, we have noted that for all real numbers x, $(x < 10) \vee (x > 0)$. So in particular, the statement $(100 < 10) \vee (100 > 0)$ is true, and so is $(5 < 10) \vee (5 > 0)$, and so is $(0 < 10) \vee (0 > 0)$. Of course, a mathematician does not *pointlessly* write down "$P \vee Q$" when he or she and the reader both know that P is true, or both know that Q is true, any more than a nonmathematician would. But it must be understood that $P \vee Q$ is *true* in such cases, in order that the truth of a mathematical statement not be lost when our knowledge increases.

Similarly, in nonmathematical usage we generally make statements of the form "If P then Q" only when there is some uncertainty as to whether the statements P and/or Q hold; but again, to make mathematical usage consistent, we must accept "$P \Rightarrow Q$" as true in all cases *except* when P holds and Q does not.

Note also that in nonmathematical usage, "P or Q" sometimes means "P or Q, but not both". (E.g., in the "John or Stephanie" example and the "which hand" example above.) In mathematical usage, the meaning of "or" is not restricted in that way; so $(5 < 10) \vee (5 > 0)$ is a true statement.

$\Leftrightarrow$, iff $\quad P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, i.e., "$P$ is true if and only if Q is true". Thus, "$\Leftrightarrow$" is synonymous with "if and only if", often abbreviated by mathematicians to "iff".

The next two exercises give some practice with the logical connectives described in this section.

Exercise 2. Suppose x is an element, and $A,\ B$ are sets. Find, for each statement in the left-hand column below, the logically equivalent statement in the right-hand column. (There is one statement in the right-hand column not equivalent to any of the statements in the left-hand column.)

$x \in A \cup B$	$(x \in A) \wedge (x \in B)$
$x \in A \cap B$	$(x \in A) \Rightarrow (x \in B)$
$x \in {}^{c}A$	$(x \in A) \wedge (x \notin B)$
$x \in A \setminus B$	$x \notin A$
	$(x \in A) \vee (x \in B)$

Exercise 3. Suppose P and Q are two mathematical assertions. (Examples might be "$n > 0$" and "n is even" if we are talking about an integer n.) Find, for each statement in the left-hand column below, the logically equivalent statement in the right-hand column. (There are two statements in the right-hand column not equivalent to any of the statements in the left-hand column.)

$P \wedge Q$	$Q \Rightarrow P$
$P \vee (\neg Q)$	$Q \wedge P$
P	$(\neg P) \Rightarrow Q$
$P \vee Q$	$(\neg P) \wedge (\neg Q)$
$\neg (P \wedge Q)$	$(\neg P) \vee (\neg Q)$
	$\neg P$
	$\neg\neg P$

General warning: Do not confuse *statements* (called by logicians *propositions*) with *sets*. For instance, if X is a statement, e.g., "$a > 10$", then it makes no sense to write "$u \in X$" or "$a \in X$". And if X and Y are sets, then "$X \Rightarrow Y$" makes no sense.

There are, of course, important relations between statements and sets. For instance, if X and Y are *sets,* then $X \subseteq Y$ is a *statement*, while if $P(x)$ is a *statement* about a real number x, then $\{x \in \mathbf{R} \mid \mathrm{P}(x)\}$ is a *set*.

3. Quantifiers

Here are two symbols that are extremely important in constructing mathematical statements.

$\forall$ "For all" or "for every". If we are referring to real numbers, then $\forall x,\ (x+1)^2 = x^2 + 2x + 1$ is a true statement; so is $\forall x,\ (x < 0) \vee (x = 0) \vee (x > 0)$. Referring to integers n, the statement $\forall n,\ (2n-1)^2 > 0$ is true, though it is not true for real numbers.

To make such formulas precise, we should show what class of possible "n" we are talking about. We can do this by writing, for instance, $\forall n \in \mathbf{Z},\ (2n-1)^2 > 0$. For greater clarity parentheses may be introduced, e.g., $(\forall n \in \mathbf{Z})\ (2n-1)^2 > 0$ or $(\forall n \in \mathbf{Z})\ ((2n-1)^2 > 0)$.

$\exists$ "There exists ... such that", or "for some". For example, $(\exists\, x \in \mathbf{Z})\ x > 10$ says that there exists an element x belonging to the set of integers, and such that $x > 10$; or briefly "There exists an integer greater than 10''. Similarly, $(\exists\, x \in \mathbf{R})\, x^2 = 3$ means ''There exists a real number whose square is 3''. (Both of these are true statements.) Still another way of reading $(\exists\, x \in \mathbf{R})$ is ''For at least one real number x it is true that''.

(Some people also write $\exists !$ or $\exists 1$ to mean ''For exactly one value'', or equivalently, ''There exists a unique value such that''; but we will not use this notation here.)

Exercise 4. Suppose $P(x)$ and $Q(x)$ are statements about an integer x. (Examples of such statements are ''$x > 0$'', ''x is odd'', ''$x \neq 55$'', ''$x = x$'', etc..) In each of the cases below, if you believe that the equivalence asked about holds, say briefly why, while if you decide that two statements are not equivalent, try to find an example of propositions P and Q for which one of the statements is true, and the other is not.

(a) Is $(\forall x)\, P(x)$ equivalent to $\neg (\exists\, x)\, (\neg P(x))$?

(b) Is $(\exists\, x)\, (P(x) \wedge Q(x))$ equivalent to $((\exists\, x)\, P(x)) \wedge ((\exists\, x)\, Q(x))$?

(c) Is $(\forall x \in \{1, 2, 3\})\, P(x)$ equivalent to $P(1) \wedge P(2) \wedge P(3)$?

(d) Is $\neg (\forall x)\, P(x)$ equivalent to $(\forall x)\, (\neg P(x))$?

Exercise 5. To show that the statement $(\exists\, x \in \mathbf{Z})\, x^2 = x$ is true, it is enough to give the single example $x = 0$. Suppose $P(x)$ is a statement about an element x, and we want to prove one of the statements below. In which cases can this be done by giving just a single example? In each such case, say what the nature of the example must be.

(a) $(\exists\, x)\, P(x)$ (c) $(\forall x)\, \neg P(x)$

(b) $(\forall x)\, P(x)$ (d) $\neg(\forall x)\, P(x)$

Exercise 6. Suppose A and B are sets. Translate each of statements (a)-(b) below, which are expressed using the symbols $\forall$ and $\exists$, into a statement about A and B expressed using only the set-theoretic symbols discussed in the first section of this note.

(a) $(\forall x)\, ((x \in A) \Rightarrow (x \in B))$ (c) $(\forall x)\, (x \notin A)$

(b) $(\forall x)\, ((x \in A) \Leftrightarrow (x \in B))$ (d) $(\exists\, x)\, ((x \notin A) \wedge (x \in B))$

Exercise 7. Suppose $A_0, A_1, A_2, \ldots$ are subsets of a set X.

(a) Match each set on the left with the set on the right that is equal to it.

$\bigcup_{i \in \mathbf{N}} A_i$ $\{x \in X \mid (\exists\, i \in \mathbf{N})\, (x \in A_i)\}$

$\bigcap_{i \in \mathbf{N}} A_i$ $\{x \in X \mid (\forall\, i \in \mathbf{N})\, (x \in A_i)\}$.

(b) Show that one has the equality $\{x \in X \mid (\exists\, i \in \mathbf{N})\, (x \in A_i)\} = \{x \in X \mid (\forall\, i \in \mathbf{N})\, (x \in A_i)\}$ if and only if the sets $A_0, A_1, A_2, \ldots$ are all equal.

Note: The English word ''any'' sometimes means $\forall$, and sometimes $\exists$; we usually understand which is meant from context. Thus, if you say, ''I wonder whether anyone knows'', you are asking whether $(\exists\, x)\, (x \text{ knows})$ is true. But the sentence ''Anyone you ask will be able to tell you'' means ''$(\forall x)$ (If you ask x, x can tell you)''. Hence in learning to use the mathematical symbols $\forall$ and $\exists$, you must pay attention to the *meanings* of statements, not just the English words used.

4. Bound and free variables

Suppose we write an equation such as $x^5 = x$. There are various things that we may mean:

(i) x may represent a definite number that we are considering, e.g., the height of a certain bridge in meters, or the greatest common divisor of $2^5 - 1$ and $2^8 - 1$. In this case, $x^5 = x$ is an assertion about that number. This assertion is either true or false.

(ii) We may regard $x^5 = x$ as a *condition* on integers x. This is then satisfied by some integers, and not by others. Taken by itself, it is neither ''true'' nor ''false''.

(iii) We may be asserting $x^5 = x$ as an *identity*. E.g., if we are considering integers, then by $x^5 = x$ we would really mean $(\forall x \in \mathbf{Z})\, x^5 = x$, which is false. On the other hand, in Math 113 one encounters the ring $\mathbf{Z}_5$, and $x^5 = x$ is a valid identity in that ring; i.e., $(\forall x \in \mathbf{Z}_5)\, x^5 = x$ is true.

(iv) We may use the equation $x^5 = x$ in the proposition $(\exists\, x \in \mathbf{Z})\, x^5 = x$ (which is true).

(v) We may use the equation $x^5 = x$ in defining the set $\{x \in \mathbf{Z} \mid x^5 = x\}$ (which equals $\{-1, 0, 1\}$).

In use (i), x is a *constant*; it represents a specific number we are talking about (even if we don't know its value), and as we have said, the statement $x^5 = x$ is then true or false.

In uses (ii)-(v), x is a *variable*. But there is a difference between (ii) and the other cases. In (ii), $x^5 = x$ is a condition in which we may substitute different values of x, making the condition true or false; x is called a *free variable*. In (iii) and (iv), the variable is *bound* by the quantifier $\forall$ or $\exists$. One cannot ask whether the statement $(\exists\, x \in \mathbf{Z})\, x^5 = x$ is "true for $x = 3$", although one can ask whether $x^5 = x$ is true for $x = 3$. Similarly, it makes no sense to ask whether $(\forall x \in \mathbf{Z})\, x^5 = x$ is true for $x = 1$ or for $x = 2$, because it is not a statement about a single integer x, but a statement whose validity is determined by substituting *all* integers for x in the statement $x^5 = x$, and seeing whether it holds in every case. (It doesn't, so as mentioned in (iii) above, "$(\forall x \in \mathbf{Z})\, x^5 = x$" is false.)

One could avoid the ambiguity of "$x^5 = x$" meaning either (i), (ii) or (iii) by insisting that different letters be used for constants and variables, and that the symbol $\forall x \in X$ be written whenever it is meant. We shall not impose such strict rules, but we should always *understand what we mean*, and be explicit *when necessary for clarity*.

A *bound variable* is an example of the more general concept of a *dummy variable*. This is a variable symbol, say x, which occurs within an expression, but such that the expression is not a function of x; rather, the value of the expression is determined by some process that involves considering different values of x. We have seen how this is so in (iii) and (iv). The x in (v) is also a dummy variable, because the set in question is determined by looking over all values of x in $\mathbf{Z}$, and collecting those which satisfy $x^5 = x$. You have seen similar situations in calculus: recall the difference between formulas like n^2 and $\Sigma_{n=1}^{10} n^2$. The first is a function of n, while the second represents a specific number, 385, computed with the help of the first function. Likewise, in the expression $\int_{x=0}^{1} x^2\, dx$, x is a dummy variable.

(There is a sixth meaning that "$x^5 = x$" can have, which one learns about in the latter half of Math 113. Under this interpretation, x is an *indeterminate* in a polynomial ring such as $\mathbf{Q}[x]$. I will say no more about this here, except that so interpreted, the equation $x^5 = x$ is false, since x^5 and x are different polynomials; and that this interpretation is like (i) in that x is a particular element rather than a variable.)

Exercise 8. (a) Give an elementary description of $\{x \in \mathbf{R} \mid (\exists\, y \in \mathbf{R})\, x < y < x^2\}$, and prove that your description indeed describes the set.

(Suggestion: First figure out by experimentation which real numbers belong to that set, then think about how to prove the answer you get. To do so, you will need to prove two sets equal, namely the set given above, and the set you describe. Two sets X and Y are equal if and only if every element of X is an element of Y, and vice versa. So to prove such an equality, one can begin "Let $r \in X$," and deduce from what is known that $r \in Y$, and then turn around and say "Now let $r \in Y$," and deduce from what is known that $r \in X$.)

(b) Give an elementary description of $\{x \in \mathbf{R} \mid (\exists\, y \in \mathbf{Z})\, x < y < x^2\}$, and again prove it is correct.

5. Order of quantifiers

If we take a sentence about integers, involving a *free* variable x, and attach to it one of the prefixes $(\exists\, x \in \mathbf{Z})$ or $(\forall x \in \mathbf{Z})$, then we have seen above that we get a new statement, in which x is a *bound* variable.

Now consider a sentence with two free variables, such as $y = x^2$, a condition on a pair of integers x and y. Suppose that we add to this the prefix $(\exists\, x \in \mathbf{Z})$, getting the statement $(\exists\, x \in \mathbf{Z})\, y = x^2$. Then x has been bound, and the result is a condition on the integer-valued variable y; in words, "y is a square". For some values of y

this is true, namely 0, 1, 4, 9, ... For other values it is false.

In particular, since the set of y for which this condition is satisfied is nonempty, it is true that $(\exists\, y \in \mathbf{Z})(\exists\, x \in \mathbf{Z})\ y = x^2$. Since the set does not contain *all* integers, it is false that $(\forall\, y \in \mathbf{Z})(\exists\, x \in \mathbf{Z})\ y = x^2$. These examples illustrate the process of adding several prefixes to a statement, so as to successively bind several variables.

Consider now the statement about two integers x and y: $x > y$. Note that the statement $(\exists\, x \in \mathbf{Z})\ x > y$ is true for all y, because there is no largest integer y. Hence

$$(\forall\, y \in \mathbf{Z})(\exists\, x \in \mathbf{Z})\ x > y$$

is a true statement. On the other hand, the statement $(\forall\, y \in \mathbf{Z})\ x > y$ is not true for any integer x; if it were, then x would be an integer larger than all integers (including itself!) Hence

$$(\exists\, x \in \mathbf{Z})(\forall\, y \in \mathbf{Z})\ x > y$$

is a false statement.

Since one statement is true and the other false, they do not mean the same thing; so a change in the *order* of the prefixes $\exists\, x \in \mathbf{Z}$ and $\forall\, y \in \mathbf{Z}$ can change the meaning of a statement!

Exercise 9. Consider the sentence, "There is someone at the hotel who cleans each room". Explain two ways this sentence can be interpreted, and translate them into two quantifications of the relation "(X cleans R)". Which words in the sentence correspond to "$\exists$" in the translation, and which to "$\forall$"?

Ambiguities in the meanings of English sentences like the one in the above exercise are generally cleared up by context. So I repeat what I said at the end of section 3: in translating a sentence into symbols, we must look at the idea, not just the words, to see how the quantifiers should be used.

Some mathematicians treat "$\forall$" simply as an abbreviation of the words "for all", and put it where they might put those words in a sentence, writing things like "$n+1 > n\ \forall n$". I strongly advise against this; under that usage, a formula $\exists\, x\, P(x, y)\ \forall y$ has exactly the ambiguity illustrated in the above exercise, since one might "bracket" it either as $\exists\, x\ (P(x, y)\ \forall y)$ or as $(\exists\, x\ P(x, y))\ \forall y$. Rather, I recommend putting quantification symbols *before* the statement being quantified, as in the examples given above.

In the next exercise, you will get practice with quantifiers by using them to write symbolically some definitions that were given with the help of words in freshman calculus.

Exercise 10. Let $f: \mathbf{R} \to \mathbf{R}$ be a real-valued function of one real variable. Translate conditions (ii)-(vi) listed below into symbols. Since all variables here are real-valued, you may omit "$\in \mathbf{R}$". Part (i) is done for you, as an example.

In your answers, do not use symbols such as $\lim_{u \to x} f(u)$, or d/dx, since these are the concepts you are trying to define. Use only basic operations and relations of the real numbers, such as $+$, $-$, $\cdot$, $|x|$, $>$, $<$, etc.. (In this exercise, don't even use in later parts concepts you have defined in earlier parts; rather, if appropriate, incorporate the symbols of the earlier translation into the later one.)

You may use letters such as ε and δ for some of your variables; but do not write things like "$\delta(x)$" to mean "a number δ which may depend on x", or, by the same token, "$f'(x)$". Rather, the order of quantifiers in your answers should show what can and what cannot depend on what.

(i) f is bounded. *Answer:* $(\exists\, M)(\forall x)\ |f(x)| < M$.
(ii) f is continuous at the point $x \in \mathbf{R}$.
(iii) f is everywhere continuous.
(iv) f has a limit as $x \to +\infty$.
(v) f is differentiable at $x = 5$.
(vi) f is everywhere differentiable.

We end with a quick self-test on material from this and preceding sections.

Exercise 11. Mark the following true or false. (Answers at the bottom of page.)

(a) $\mathbf{N} \in \mathbf{Z}$.
(b) $\mathbf{N} \subseteq \mathbf{Z}$.
(c) $\mathbf{Z} \cup \mathbf{Q} = \mathbf{Q}$.
(d) $\mathbf{Z} \cap \mathbf{N} = \mathbf{N}$.
(e) If subsets A and B of a set X satisfy $A \cap B = \varnothing$, then $A = X - B$.
(f) $(\exists\, x \in \mathbf{Q})\ (\forall y \in \mathbf{Q})\ \ x = y$.
(g) $(\exists\, x \in \mathbf{Q})\ (\forall y \in \mathbf{Q})\ \ x \neq y$.
(h) $(\forall x \in \mathbf{Q})\ (\exists\, y \in \mathbf{Q})\ \ x = y$.
(i) $(\forall x \in \mathbf{Q})\ (\exists\, y \in \mathbf{Q})\ \ x \neq y$.
(j) $(\exists\, x \in \mathbf{Q})\ (\forall y \in \mathbf{Z})\ \ x \neq y$.

6. Some mathematical language

There are several turns of phrase used in mathematical writing that one generally picks up by seeing how they are used. But it can be helpful to have explanations available, so I give below (at the suggestion of John Peloquin) a glossary of some of these phrases.

necessary and sufficient conditions. To say that a statement A is necessary and sufficient for a statement B to hold simply means that $A \Longleftrightarrow B$. For instance, if x is a real number, a necessary and sufficient condition for $\lim_{n\to\infty} x^n = 0$ to hold is that $|x| < 1$.

In this usage, "sufficient" refers to the forward implication $A \Rightarrow B$, and "necessary" to the reverse implication $A \Leftarrow B$. These words can also be used separately. So, in considering whether a sum of integers $m + n$ is even, we see that a *sufficient* condition is that both m and n be even, but it is not necessary; while for $m + n$ to be odd, a *necessary* condition is that at least one of m and n be odd, but it is not sufficient. When one proves a statement of the form $A \Longleftrightarrow B$ by proving the implication first in one, and then in the opposite direction, the proof that $A \Rightarrow B$ is often called the proof of *sufficiency*, and the proof that $A \Leftarrow B$ the proof of *necessity*.

A necessary and sufficient condition for something to be true is called a ***criterion*** for it to be true; one also speaks of "necessary criteria" and "sufficient criteria", which just go one way. For instance, most of the tests for convergence of a series Σa_n that one learns in calculus are *sufficient criteria* for convergence; but the statement that for Σa_n to converge one must have $\lim_{n\to\infty} a_n = 0$ is a *necessary* criterion.

If one proves a necessary and sufficient condition for an element x to have a certain property, this is called a ***characterization*** of the elements that have that property.

to identify. To "identify" two mathematical objects means to regard them as the same. For instance, when we consider the geometry of the plane, $\mathbf{R}^2 = \{(x, y) \mid x, y \in \mathbf{R}\}$ and of three-dimensional space $\mathbf{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbf{R}\}$, we often regard $\mathbf{R}^2$ as a subset of $\mathbf{R}^3$, by identifying each point (x, y) of the plane with the point $(x, y, 0)$ of 3-space; we thus *identify* $\mathbf{R}^2$ with the (x, y)-plane $\{(x, y, z) \mid z = 0\}$ in $\mathbf{R}^3$.

How one justifies regarding two different things as the same, in a precise logical science such as mathematics, takes some pondering. In examples like the above, it can be thought of as a notational shorthand; we can say that when we speak about points and subsets of the plane $\mathbf{R}^2$ as lying in 3-dimensional space, we do not mean those points themselves, but their images under the map $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^3$ defined by $\varphi((x, y)) = (x, y, 0)$; and because φ preserves geometric structure (e.g., distance, the property of lying in a straight line, etc.), geometric statements about points of $\mathbf{R}^2$ remain true of their images under φ.

Some other uses of the term are a little different. For instance, the unit circle, parametrized by radian measure, is sometimes described as "the interval $[0, 2\pi]$ with the two ends 0 and 2π identified". I will not go into how to think of this sort of identification here.

well-defined. When one gives a statement that is supposed to be a definition, it is sometimes necessary to verify that it really does precisely define some mathematical object. For instance, if we tried to define a function

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Answers to Exercise 11: (a) F. (b) T. (c) T. (d) T. (e) F. (f) F. (g) F. (h) T. (i) T. (j) T.

a from positive rational numbers to positive rational numbers by saying that whenever $r = m/n$ with m and n positive integers, we let $a(r) = (m+1)/(n+1)$, we would face the problem that a positive rational number can be written in more than one way as a ratio of positive integers; e.g., $2/3 = 4/6$; so we would need to know whether our definition depended only on the given rational number r, or on the way we chose to write it as a ratio m/n. In fact, we see that $(2+1)/(3+1)$ and $(4+1)/(6+1)$ are *not* the same rational number; so the above is not a usable definition. On the other hand, if for $r = m/n$ we define $b(r) = m^2/n^2$, we find that this rational number does *not* depend on our choice of how to write r as a ratio. (In fact, $b(r) = r^2$.) If some entity we have defined is indeed determined by the rules we have stated, rather than varying with choices implicit in our definition, we say that it is *well-defined*.

Proofs of well-definedness become essential in areas of mathematics where certain entities are defined as *equivalence classes* of other entities, and operations on them are defined by choosing "representatives" of these equivalence classes, performing operations on these representative, and taking the equivalence class of the resulting element. This is not the place to go into those constructions; but you will see proofs of well-definedness coming up frequently when you study such topics.

unique. The *unique* element having a property means the *only* element with that property. (Thus, in mathematics, the word "unique" is always used relative to the statement of some property – often mentioned in the same sentence, but sometimes implicit in the context.) For instance, 2 is the unique real number x such that $x^3 = 8$; so that equation has a *unique solution* in $\mathbf{R}$. On the other hand, the equation $x^2 = 4$ does not have a unique solution in $\mathbf{R}$, since both 2 and -2 are solutions.

(I have actually used the word "unique" four times in the preceding pages of this note, trusting that most students either knew its mathematical meaning, or would recognize what I meant.)

up to ... . This phrase allows one to modify a statement so as to allow more leeway. For instance, if a is a positive real number, then the equation $x^2 = a$ has a real-number solution which is *unique up to sign*. This means that if x_1 and x_2 are both solutions to that equation, we do not assert that $x_1 = x_2$ (as we would if we simply said that the solution was unique), but, rather, that $x_1 = \pm x_2$. Likewise, if $f(x)$ is a continuous function on the real line, then there is a function g whose derivative is f, and this g is *unique up to* (or *determined up to*) an additive constant. In High School geometry, when one learns that a triangle is "determined by side-angle-side", i.e., by the lengths of two sides and the value of the angle between them, a fuller statement would be that the triangle is determined *up to congruence* by this data. In other words, triangles that agree in this data must be congruent, but need not actually consist of the same points of the plane. (The fact that most geometry textbooks do *not* add "up to congruence" to such statements means that in these statements, they are *identifying* congruent triangles.)

without loss of generality. In giving a mathematical proof, if we say that "without loss of generality" we may assume that some condition X holds, this means that *if* we can establish the result in the case where X holds, we can deduce from this that it holds in general. After saying this, one usually assumes that X holds for the rest of the proof.

For instance, in proving a theorem about a function f on a closed interval $[a, b]$, an author might say, "Without loss of generality, we may assume $[a, b] = [0,1]$". Typically, the reason is that if f is a function on $[a, b]$, then the function g on $[0,1]$ defined by $g(x) = f(a+(b-a)x)$ has properties closely corresponding to those of the original function f. (For instance, $g(0) = f(a)$, $g(1) = f(b)$, g is differentiable if and only if f is differentiable, etc..) Depending on the theorem one is trying to prove, one may be able to see that knowing the theorem is true for the above function g implies that it is true for f. In that case, it suffices to go through the details of one's proof for functions on $[0,1]$; and, if this makes the proof easier to write out or to follow, one may say, "Without loss of generality, we shall assume $[a, b] = [0,1]$", and complete the proof under that assumption.

Of course, whether it is "clear" that knowing a result in one case implies that it is true in other cases depends on the situation, and on the mathematical background of one's readership. If the author of a text you are reading says, without further explanation, that without loss of generality some assumption may be made, this means that he or she judges that the reduction to that case should be straightforward for students at the level at which the text is aimed; and you should take up the challenge, and see whether you can supply the reason. If you can't, you should ask your instructor. In other cases, an author may say explicitly *why* a "without loss of generality" statement is justified. You should then look carefully at the arguments by which he or she reduces the general case to the special case.

(Mathematicians writing for other mathematicians often abbreviate "without loss of generality" to "w.l.o.g."; but this abbreviation seldom appears in undergraduate textbooks.)

The turns of phrase listed above are ones that I have seen students have a great deal of trouble with. The next couple haven't led to problems as often, but they are also worth noting.

maximal. This is a term that is used in the context of sets that have among their members a relation of some being "greater than" others. This is not the place to discuss the various ways in which such relations arise, so I will just talk about one case: sets, with the "greater than" relation being the relation of one set having the other as a subset.

So suppose S is some set of subsets of a set X. Then an element $A \in S$ is said to be *maximal* in S if no *other* member of S has A as a subset.

For instance, if we take $X = \{1, 2, 3, 4, 5\}$, and let S consist of all subsets of X that do not contain any two adjacent integers (integers that are "next to" each other in the list $1, 2, 3, 4, 5$), then $\{1\} \subsetneqq \{1, 3\} \subsetneqq \{1, 3, 5\}$ are members of S, and these inclusions imply that $\{1\}$ and $\{1, 3\}$ are *not* maximal elements of S. You might check for yourself that S has exactly four maximal elements: $\{1, 3, 5\}$, $\{1, 4\}$, $\{2, 4\}$ and $\{2, 5\}$.

An element in such a set which contains all other elements is called a ***greatest*** element of the set. If a set has a greatest element, that will also be a maximal element, but as the example of the preceding paragraph shows, not every maximal element is a greatest element; the set S of that paragraph does not have a greatest element. An example of a set that has no maximal elements (and hence also no greatest element) is the set of all finite subsets of $\mathbf{N}$.

Reversing the order-relations in the above discussion gives the concepts of ***minimal*** elements and ***least*** elements.

by choice of ... This is best illustrated by an example. If in an argument one has said "Suppose the polynomial $f(x)$ has a positive root r", then if one later says that something is true "by choice of r", this means it is true because r is a root of $f(x)$, or because r is positive, or because both these statements are true; in other words, because of one or more assumptions we made when we specified r. So the phrase "by choice of..." is a signal to look back at the point where an element was introduced, and see what was assumed about it.

Let me end with a **warning** about an incorrect use of words I have often seen students make. If one wants to describe $\{n^2 \mid n \in \mathbf{Z}\}$, it is not correct to call this "the set containing all squares of integers", because there are many sets that fit those words. For instance, the set of all integers, and the set of all real numbers, both *contain* all squares of integers (along with other elements). The correct description of $\{n^2 \mid n \in \mathbf{Z}\}$ is "the set of all squares of integers". If, for some reason, one wants a more explicit word than "of", one may say "the set consisting of all squares of integers", or "the set whose members are all squares of integers".