Exercises supplementing those in W. Boyce and R. DiPrima's

***Elementary Differential Equations and Boundary Value Problems*, 7th edition**

This is a collection of exercises that I have put on homework sheets for the differential equations part of Math 54M (and its predecessor Math 50A) over the years, supplementing the exercises in Boyce and DiPrima. They follow the text's notation, and range from exercises that are comparable in difficulty to those in the text but fill some gap or give some interesting perspective on the material, through exercises that would be ''challenge problems'' in a regular course, and good exercises for an honors courses. Some of those of moderate difficulty I assign as part of the homework. Others of that sort, and all of the more difficult problems, along with some exercises from the text, I recommend for ''students interested in further interesting and/or more challenging problems'' to try on their own.

I have given the exercises numbers indicating the relevant section of Boyce and DiPrima. If an exercise was closely related to exercise n in a given section of the text, I have generally called it ''Exercise $n+½$'' of that section; I give the other exercises numbers continuing where the numbers in the text leave off. There are a large number of the former sort because Boyce and DiPrima show solutions to all computational exercises in the back, and I wanted homework problems whose answers the students would have to work out for themselves.

The exercises below correspond to sections of the book covered in the courses I have taught. Some years ago, a large amount of basic differential equations was moved out of this course and into the second-Semester calculus course (taught from Stewart) leaving only about six weeks of differential equations in the course taught using Boyce and DiPrima. Hence on occasions since then when I have taught this course I have not given exercises relevant to the sections of Boyce and DiPrima that were removed. However, I have also included below a few exercises from those sections which I gave in the earlier course, with notation and chapter-reference adjusted to the current edition of the text.

Since the course begins with linear algebra, when we come to differential equations we note that the existence and uniqueness theorem for an nth order linear differential equation gives us, in the homogeneous case, the dimension of the space of solutions. Although this is not discussed in the text, one exercise below (§3.2, ''Exercise 38'') concerns this topic.

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§1.3, ''Exercise 30'': Suppose p_1, p_2, φ_1, φ_2 are functions such that $\varphi_1(t)$ is a solution to the differential equation $y' + p_1(t)\,y = 0$, and $\varphi_2(t)$ is a solution to the differential equation $y' + p_2(t)\,y = 0$. Show that $\varphi_1(t)\,\varphi_2(t)$ is a solution to $y' + (p_1(t) + p_2(t))\,y = 0$.

§3.1, ''Exercise 44'': You have seen in §3.1 how to solve the equation $ay'' + by' + cy = 0$ when the characteristic equation $ar^2 + br + c = 0$ has distinct real roots. Now a quadratic equation with a *double* root can be thought of as a limit of quadratic equations with distinct roots, as these roots get closer and closer together. This suggests that one might discover how to solve a differential equation whose characteristic equation has a double root r by taking a limit of solutions to the differential equations whose characteristic equations have roots r and $r+\varepsilon$, as $\varepsilon \to 0$. This is the idea developed below.

(a) If r and ε are real numbers, determine the real numbers $b_{r,r+\varepsilon}$ and $c_{r,r+\varepsilon}$ such that the quadratic equation

$$t^2 + b_{r,r+\varepsilon}\,t + c_{r,r+\varepsilon} = 0$$

has r and $r+\varepsilon$ as its roots. (Hint: if the polynomial has these two roots, how must it factor?)

(b) For the $b_{r,r+\varepsilon}$ and $c_{r,r+\varepsilon}$ computed above, find the general solution to the differential equation

$$y'' + b_{r,r+\varepsilon}\,y' + c_{r,r+\varepsilon}\,y = 0,$$

assuming $\varepsilon \neq 0$. (Hint: you already know the roots of its characteristic equation.)

(c) Find the solution to the above differential equation that satisfies the initial conditions $y(0) = 0$,

$y'(0) = 1$. We shall call this function $u_{r,r+\varepsilon}$.

(d) Holding r fixed and letting ε go to zero, compute $\lim_{\varepsilon\to 0} u_{r,r+\varepsilon}(x)$. Call this function $u_{r,r}(x)$.

(e) Write down the polynomial $t^2 + b_{r,r}\,t + c_{r,r}$, and test whether $u_{r,r}$ satisfies the differential equation

$$(*)\qquad y'' + b_{r,r}\,y' + c_{r,r}\,y \;=\; 0.$$

(f) Name a nonzero function v that is a common solution to the equations $y'' + b_{r,r+\varepsilon}\,y' + c_{r,r+\varepsilon}\,y = 0$ for all $\varepsilon \neq 0$. (Hint: what root do the characteristic equations of all these differential equations have in common?) Verify that this v is also a solution to the equation (*).

(g) What is the most general solution to the equation (*) that you can get by combining the above two solutions?

(h) As an application, find the solution to the differential equation $y'' - 2y' + y = 0$ satisfying the initial conditions $y(1) = 1$, $y'(1) = 0$.

§3.2, ‘‘Exercise 6½’’: Find the Wronskian of each of the following pairs of functions:

(a) x^2, x^4. (b) $\sin x$, $1 + \sin x$. (c) e^x, e^{x+c}.

§3.2, ‘‘Exercise 12½’’: Same instructions as for Boyce and DiPrima’s Exercises 7-12 for the equation

$$(t^2 - 100)\,y'' + t(t^2+1)^{-1}y' + (t^2-9)(t^2-25)^{-1}y \;=\; (t+1)/(t-1),$$

for each of the initial conditions

(a) $y(0) = y'(0) = 0$, (b) $y(4) = -1$, $y'(4) = 1$, (c) $y(15) = 10^{10}$, $y'(15) = 10^{10^{10}}$.

§3.2, ‘‘Exercise 19½’’: If $W(f,g)$ is the Wronskian of f and g, and if $u = af + bg$, $v = cf + dg$ for real constants a, b, c, d, find the Wronskian $W(u,v)$ in terms of $W(f,g)$.

§3.2, ‘‘Exercise 38’’: Let S be a subset of the real line of the form $(a,b) \cup (c,d)$ where $a < b < c < d$ are real numbers. That is, S consists of all points in (a,b) together with all points in (c,d).

(a) Show that for all real numbers c_1 and c_2, the function on S whose value on the interval (a,b) is everywhere equal to c_1, and whose value on (c,d) is everywhere equal to c_2, is a solution on S to the differential equation $y' = 0$.

(b) Conclude that the set of solutions to that differential equation on S is a 2-dimensional vector space. (We recall, in contrast, that the set of solutions of a first order homogeneous linear differential equation on an *interval* is a 1-dimensional vector space.)

(c) Show, more generally, that if $y' + p(t)\,y = 0$ is a homogeneous first order linear differential equation with $p(t)$ continuous at all points of S, then the set of solutions to this equation on S forms a 2-dimensional vector space.

(d) Likewise, show that if $y'' + p(t)y' + q(t)y = 0$ is a homogeneous second order linear differential equation with $p(t)$, $q(t)$ continuous at all points of S, then the set of solutions to this equation on S forms a 4-dimensional vector space. (You can assume facts stated in the reading.)

§3.3, ‘‘Exercise 21½’’: For y_1, y_2 as in Exercise 21, find $W(y_1, y_2)(5)$.

§3.3, ‘‘Exercise 29’’: We have seen that if the Wronskian of two functions f and g is not everywhere zero, these functions are linearly independent, while in the contrary case, f and g *tend* to be linearly dependent; though §3.3, Exercise 28 (p.153) shows that this is not always so. In this exercise, you can obtain a result showing that in a large class of cases, if the Wronskian is zero the functions are indeed linearly dependent.

Suppose f and g are differentiable functions on an open interval I, and that $W(f,g)$ is 0 for all x in I.

(a) Compute $(f/g)'$, and show that (at those points of I for which your computation is meaningful and valid), the result is 0.

(b) At what points of I is the above computation meaningful and valid?

(c) Deduce that, under an appropriate assumption, f and g are linearly dependent.

§3.3, ‘‘Exercise 30’’: If f and g are solutions to a differential equation $y'' + p(t)y' + q(t)y = 0$, we know that their Wronskian is either nowhere zero or everywhere zero. But for general functions f and g the Wronskian may be a nonzero function which has *some* zeroes.

This exercise will show that when the Wronskian has no zeroes (as in the situation mentioned in the first sentence if the functions are linearly independent), this puts interesting restrictions on the behavior of the functions.

Suppose f and g are differentiable functions on an open interval I.

(a) Show that if f has two zeroes $x_0 < x_1$ in I (i.e., if $f(x_0) = f(x_1) = 0$), and if g has no zeroes in $[x_0, x_1]$, then there is a point between x_0 and x_1 where the Wronskian of f and g is 0. (Hint: Using a result from first year calculus, show that there is a point where the derivative of f/g is 0; then use the calculation of part (a) of ‘‘§3.3, Exercise 29’’.)

(b) Deduce that if the Wronskian of f and g is nowhere 0 on the interval I, then between every two zeroes of f there is a zero of g.

(c) Deduce the result of §3.4, Exercise 33, p.159. (The final sentence of that exercise, ‘‘Note that ...’’, is not something you are asked to write out; it is just an illustration of the result you have proved.)

§3.4, ‘‘Exercise 22½’’: Like exercises 17-22, but for the equation $y'' - 4y' + 5.21y = 0$ and initial values $y(5\pi) = 1$, $y'(5\pi) = 1$.

§3.4, ‘‘Exercise 43’’: Using the formulas obtained in Exercise 29 (p.159), verify the following trigonometric identities:

(a) $\sin^2 x + \cos^2 x = 1$ (b) $\sin 2x = 2\sin x \cos x$ (c) $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$.

§3.4, ‘‘Exercise 44’’: Prove, for an appropriate choice of constant k (which you will discover in the course of the computation), the trigonometric identity $\cot 2x = k(\cot x - \tan x)$, by rewriting each side of this equation in terms of complex exponentials with the help the formulas obtained in Exercise 29 (p.159), and simplifying the right-hand side. (You can make the computations less messy by writing $t = e^{ix}$, if you are careful about how e^{-ix}, e^{2ix}, etc. are expressed in terms of t.)

§3.4, ‘‘Exercise 45’’: For real numbers a and b, find indefinite integrals of $e^{ax}\sin bx$ and $e^{ax}\cos bx$ by the following two methods:

(a) Differentiate $Ce^{ax}\sin bx + De^{ax}\cos bx$, and find the values of C and D that give the desired functions.

(b) Substitute into $e^{ax}\sin bx$ and $e^{ax}\cos bx$ the formulas for sine and cosine in terms of complex exponentiation (§3.4 Exercise 29, p.159); integrate the expressions you get as linear combinations of exponential functions; separate the results into summands involving real and imaginary coefficients, and translate each of these into appropriate products of real exponential and trigonometric functions.

§3.7, ‘‘Exercise 33’’: (a) For c an arbitrary positive real number, solve the initial value problem

$$y'' + y = \sin ct, \quad y(0) = 0, \quad y'(0) = 0.$$

(There will be one value of c for which your solution will have a different form from other values.)

(b) For what values of c is the solution you got in part (a) *bounded* as a function of t? (To say y is bounded means that there is some constant M such that $-M \le y(t) \le M$ for all t.)

For every c that makes the solution bounded, find a bound for the solution (an M with the above properties).

§4.1, ‘‘Exercise 29’’: This exercise concerns the fact referred to in the *Hint* that the authors give to §4.1 Exercise 20(a). That fact, stated for $n\times n$ matrices and written out explicitly, says that if we are given an $n\times n$ matrix $\mathbf{A}$ of functions, with rows $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and we write D for the operator d/dt, and understand D applied to a row vector of functions to act by differentiating each component, then

$$D\left(\det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}\right) = \det \begin{bmatrix} D\mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} + \det \begin{bmatrix} \mathbf{a}_1 \\ D\mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} + \ \ldots\ + \det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ D\mathbf{a}_n \end{bmatrix}.$$

(a) Prove the above formula.

(b) Using that formula, verify the analog of the expression for W' given in §4.1 Exercise 20(a) for the case where W is the Wronskian of n functions $y_1, \ldots, y_n$. (The exercise in Boyce and DiPrima assumes y_1, y_2, y_3 are solutions of a third order linear homogeneous differential equation; but that assumption is not needed till part (b) of that exercise. All one needs is that the y’s have enough derivatives to make the formula asked for meaningful.)

(c) For W as above, use the same formula to obtain expressions for W'' and W'''. (These will be a little more complicated than the formula for W', but not very much.)

§4.1, ‘‘Exercise 30’’: Verify that a solution to the differential equation $\sin t \cos t\, y' - y = 0$ is $y = \tan t$, and that a solution to $\sin t \cos t\, y' - y = 1$ is $y = -1$. Now find the general solution to $\sin t \cos t\, y' - y = 1$ on the interval $(0, \pi)$, and the particular solution satisfying the initial condition $y(\pi/4) = 1$.

§4.1, ‘‘Exercise 31’’: Verify that a solution to the differential equation $x^2 y'' + xy' - y = x^3$ is $x^3/8$, while the equation $x^2 y'' + xy' - y = 0$ has solutions x and x^{-1}. Now find the general solution to the equation $x^2 y'' + xy' - y = x^3$ for $x > 0$, and the particular solution satisfying $y(1) = y'(1) = 0$.

§4.1, ‘‘Exercise 32’’: Verify that a solution to the differential equation $y'' + y = t^2$ is $y = t^2 - 2$. Find the solution to the same equation that satisfies the initial conditions $y(0) = a$, $y''(0) = b$ for real numbers a and b.

§4.1, ‘‘Exercise 33’’: Verify that a solution to the differential equation $y'' - 2y' + 2y = t$ is $y = (t+1)/2$. Find the solution to the same equation that satisfies the initial conditions $y(1) = 1$, $y'(1) = 2$.

§4.1, ‘‘Exercise 34’’: Let $y' + p(t)y = g(t)$ be a first order linear differential equation, with $p(t)$ and $g(t)$ continuous on the interval I, and let $y_1(t)$ and $y_2(t)$ be two distinct solutions to this differential equation. Prove that the general solution to the equation is given by $cy_1(t) + (1-c)y_2(t)$. (Hint: From the two given solutions to the above nonhomogeneous equation, how can you find a solution to the corresponding homogeneous equation? After doing so, apply the method discussed in the text for getting the general solution to a nonhomogeneous equation.)

§5.1, ‘‘Exercise 29’’: Suppose that a power series $\Sigma_{n=0}^{\infty} a_n x^n$ satisfies the equation

$$x^2(\Sigma_{n=0}^{\infty} a_n x^n) - x(\Sigma_{n=0}^{\infty} a_n x^n) - (\Sigma_{n=0}^{\infty} a_n x^n) = x.$$

In the first two expressions, bring the x^2 or x into the summation, shift the index of summation so that each term again involves x to the power n, and then express the above equation as a set of conditions on the coefficients a_n. Use these conditions to find $a_0, \ldots, a_{10}$.

§7.1, ‘‘Exercise 5½’’: Consider the initial value problem $u''' + p(t)\,u'' + q(t)\,u' + r(t)\,u = g(t)$, $u(0) = a$, $u'(0) = b$, $u''(0) = c$. Transform this into an initial value problem for three first order equations.

§7.3, ‘‘Exercise 34’’: (i) Suppose a real $n \times n$ matrix $\mathbf{A}$ has a non-real eigenvalue $\lambda + \mu i \in \mathbb{C}$. Let $\mathbf{x} + i\mathbf{y}$ be an eigenvector for this matrix, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(ii) Conversely, let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be any two linearly independent vectors, and $\lambda + \mu i \in \mathbb{C}$ any complex number with $\mu \neq 0$. Show that there is a *real* matrix $\mathbf{A}$ which has $\mathbf{x} + i\mathbf{y}$ as an eigenvector, with eigenvalue $\lambda + \mu i$. (The easiest case is $n = 2$; then $\mathbf{A}$ is uniquely determined, and so can be computed exactly from the given information. For $n > 2$ one has to make certain arbitrary choices in getting $\mathbf{A}$.)

§7.5, ‘‘Exercise 17½’’: Like Exercise 17, but with initial condition $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

§7.6, ‘‘Exercise 10½’’: (a) Find the general solution to the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} -4 & 2 \\ -1 & -2 \end{bmatrix} \mathbf{x}.$$

(b) Find the particular solution to this equation satisfying $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Describe the behavior of this solution as $t \to +\infty$.

§7.6, ‘‘Exercise 24½’’: For the equation given in §7.6, Exercise 24 (p.391), find the solution satisfying the initial condition $\mathbf{x}'(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and sketch the images of the trajectory thereof in the $x_1 x_2$-plane, in the $x_1 x_3$-plane, and in the $x_2 x_3$-plane.

§7.7, ‘‘Exercise 12½’’: For the system of equations $\mathbf{x}' = \begin{bmatrix} -6 & 4 \\ 1 & -6 \end{bmatrix} \mathbf{x}$,

(a) Find a fundamental matrix $\boldsymbol{\Psi}(t)$.

(b) Find a fundamental matrix $\boldsymbol{\Phi}(t)$ which satisfies $\boldsymbol{\Phi}(0) = \mathbf{I}$.

(c) Find a fundamental matrix $\boldsymbol{\Theta}(t)$ which satisfies $\boldsymbol{\Theta}(1) = \mathbf{I}$.

(d) Solve the initial value problem $\mathbf{x}'(t) = \begin{bmatrix} -6 & 4 \\ 1 & -6 \end{bmatrix} \mathbf{x}(t)$, $\mathbf{x}(1) = \begin{bmatrix} 1 \\ 100 \end{bmatrix}$ by using the fundamental matrix found in part (c).

§7.7, ‘‘Exercise 18’’: If $\mathbf{A}$ is an $n \times n$ matrix and $\boldsymbol{\Psi}(t)$ is a fundamental matrix for the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, show that *every* fundamental matrix for this system can be written $\boldsymbol{\Psi}(t)\mathbf{B}$ for some invertible $n \times n$ matrix of constants $\mathbf{B}$, and conversely, that for every invertible $n \times n$ matrix of constants $\mathbf{B}$, the matrix $\boldsymbol{\Psi}(t)\mathbf{B}$ is a fundamental matrix for that system.

§7.7, ‘‘Exercise 19’’: Compute the matrices $\exp\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\exp\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\exp\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. (Although the text only talks about exponentiation of matrices of the form $\mathbf{A}\,t$, the same formula can be applied to any matrix, whether or not it involves a factor ‘‘t’’.) Verify that

$$\exp\begin{bmatrix}1&0\\0&0\end{bmatrix}\exp\begin{bmatrix}1&1\\0&0\end{bmatrix} \neq \exp\begin{bmatrix}2&1\\0&0\end{bmatrix} \neq \exp\begin{bmatrix}1&1\\0&0\end{bmatrix}\exp\begin{bmatrix}1&0\\0&0\end{bmatrix}.$$

Thus, exponentiation of matrices does not satisfy the law $\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{A}+\mathbf{B})$. Contrast this with the result of Exercise 14, p.400.

(One can show that matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy $\exp(\mathbf{A})\exp(\mathbf{B}) = \exp(\mathbf{A}+\mathbf{B})$ *if* $\mathbf{A}$ and $\mathbf{B}$ commute, i.e., if $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$. This is why the result of Exercise 14 holds.)

§7.8, ‘‘Exercise 10½’’: (a) Find the general solution to the system of differential equations

$$\mathbf{x}' = \begin{bmatrix}1&1\\0&1\end{bmatrix}\mathbf{x}.$$

(b) Find the particular solution to this equation satisfying $\mathbf{x}(0) = \begin{bmatrix}0\\1\end{bmatrix}$.

(c) Determine the equation of the trajectory of this solution. (Suggestion: Solve for t as a function of one of x_1 or x_2, then substitute the resulting expression for t into the equation describing the other variable, x_2 or x_1, in terms of t. This will give an equation relating x_1 and x_2.)

§10.1, ‘‘Exercise 4½’’: Same instructions as §10.1, Exercises 1-10 (p.547) for the problem $y'' + y = 0$, $y(L) = 0$, $y'(M) = 1$ where $L \neq M$.

§10.1, ‘‘Exercise 16½’’: Same instructions as §10.1, Exercises 11-16 (p.547) for the problem $y'' + \lambda y = 0$, $y(L) = 0$, $y'(M) = 0$ where $L \neq M$.

§10.1, ‘‘Exercise 18’’: Let $y'' + p(t)y' + q(t)y = 0$ be a differential equation defined on an interval I, let y_1, y_2 be a fundamental system of solutions, and let t_0, t_1 be two points of I. Show that the following three conditions are equivalent:

(a) The matrix $\begin{bmatrix}y_1(t_0) & y_2(t_0)\\ y_1(t_1) & y_2(t_1)\end{bmatrix}$ is nonsingular.

(b) The only solution to the given differential equation satisfying $y(t_0) = y(t_1) = 0$ is the function $y = 0$.

(c) For every pair of real numbers a, b there is a unique solution to the given differential equation satisfying $y(t_0) = a$, $y(t_1) = b$.

§10.1, ‘‘Exercise 19’’: In the situation of the preceding exercise, if the equivalent conditions (a)-(c) hold, show that there exists a pair of solutions $y^{(0)}(t)$, $y^{(1)}(t)$ such that for every pair of real numbers a, b, the solution to the given differential equation satisfying $y(t_0) = a$, $y(t_1) = b$ is given by $y(t) = a\,y^{(0)}(t) + b\,y^{(1)}(t)$.

§10.2, ‘‘Exercise 8½’’: Determine whether each of the following functions is periodic. If so, give a period (if possible, the fundamental period). Briefly justify your answers. (Detailed proof not required.)

(a) $\cos x/5$. (b) $\sin(x^2)$. (c) $(\sin x)^2$. (d) $\tan 5\pi x/4$. (e) $\sin x + \cos 3x/4$.

§10.2, ‘‘Exercise 18¼’’: Same instructions as for §10.2, Exercises 13-18, for the function

$$f(x) = \sin x \qquad 0 \le x < \pi/2, \qquad f(x+\pi/2) = f(x).$$

§10.2, ‘‘Exercise 18½’’: Same instructions as for §10.2, Exercises 13-18, for the function

$$f(x) = e^x \qquad a \le x < b, \qquad f(x+(b-a)) = f(x),$$

where $a < b$ are arbitrary real numbers. (Each term of the answer will be a function of a and b. For the integration technique, see ‘‘§3.4, Exercise 44’’ above.)

§10.2, ‘‘Exercise 30’’: Show that if a function f is periodic with period T, then the function g defined by $g(x) = f(x) + f(x+T/2)$ is periodic with period $T/2$.

§10.2, ‘‘Exercise 31’’: Suppose $f(x)$ is a function having period T, and Fourier series

$$a_0/2 + \Sigma_{m=1}^{\infty} \left(a_m \cos(2\pi m x/T) + b_m \sin(2\pi m x/T)\right).$$

(a) Determine the Fourier series of the function $h(x) = f(x+T/2)$. (Assume it is legitimate to make a change of variables in each term of the Fourier series; then rewrite the result as a new Fourier series, using elementary properties of the sine and cosine.)

(b) Determine the Fourier series of the function $g(x) = f(x) + f(x+T/2)$.

(c) Show that the result you got in (b) agrees with what you proved in ‘‘Exercise 30’’ above.

§10.3, ‘‘Exercise 6½’’: For each of the functions given in §10.2, Exercises 13, 14, 16, ‘‘18¼’’ and ‘‘18½’’, sketch the graph of the function to which the series converges for three periods.

§10.4, ‘‘Exercise 6½’’: Same instruction as for Exercises 1-6, for the function $f(x) = 2/(e^x - 1) + 1$. If your answer is ‘‘neither’’, you need to show this explicitly. (For instance, to show the function is not even, you need to give a value of x at which you can show that $f(x) \neq f(-x)$, perhaps by showing that one side is positive and the other negative, or that one side is < 2 and the other ≥ 2.)

§10.4, ‘‘Exercise 22¼’’: Same instructions as for Exercises 15-22, for the function given by $f(x) = x - x^2$ for $0 < x < 1$, sine series, period 2.

§10.4, ‘‘Exercise 22½’’: Same instructions as for Exercises 15-22, for the function given by $f(x) = \sin x$ for $0 < x < \pi/2$, cosine series, period π.

§10.4, ‘‘Exercise 41’’: Let f and g be real-valued functions defined for all real numbers. Let h be the *composite* of f and g, i.e., the function defined by $h(x) = f(g(x))$.

(a) Show that if g is an even function, then so is h, no matter what the function f is.

(b) Show that if g is an odd function, then h will be even if f is even and odd if f is odd.