Mathematical Induction

1. The idea, starting from an example.

Let me introduce the idea of Mathematical Induction using a bit of math that you already know. Imagine, however, that you were discovering it for the first time.

Suppose you and a friend start calculating the sums of reciprocals of powers of 2:

$$\begin{aligned} 1/2 &= 1/2, \\ 1/2 + 1/4 &= 3/4, \\ 1/2 + 1/4 + 1/8 &= 7/8, \\ 1/2 + 1/4 + 1/8 + 1/16 &= 15/16. \end{aligned}$$

You notice a pattern: In each of the above cases,

(1) $$1/2 + 1/4 + 1/8 + \ldots + 1/2^n = (2^n - 1)/2^n.$$

The two of you wonder whether this will continue to be true for all positive integers n. Your friend spends a few days checking this all the way up to $n = 100$, adding the terms up for each case, and finds it *does* work in all these cases. Then your friend says, ''I'm tired. You take over!''

You could add up the 101 terms for the next case, $1/2 + 1/4 + 1/8 + 1/16 + \ldots + 1/2^{101}$. But it occurs to you: Most of that work is unnecessary. Your friend has already added up the first 100 of these terms. All you have to do is add on the next one. And you find that quite easy algebraically:

$$\begin{aligned} & 1/2 + 1/4 + \ldots + 1/2^{101} \\ = \ & (1/2 + 1/4 + \ldots + 1/2^{100}) + 1/2^{101} && \text{(bringing together the 100 terms your friend has added up)} \\ = \ & (2^{100} - 1)/2^{100} + 1/2^{101} && \text{(using the result of your friend's calculation).} \end{aligned}$$

Writing the above to the common denominator $1/2^{101}$, you get

$$\begin{aligned} &= (2^{101} - 2)/2^{101} + 1/2^{101} \\ &= (2^{101} - 2 + 1)/2^{101} \\ &= (2^{101} - 1)/2^{101}. \end{aligned}$$

So you've saved yourself a long, step-by-step addition, and verified that equation (1) also holds for $n = 101$.

What about the next case? Obviously, rather than starting from scratch, you can similarly make use of the $n = 101$ case. You do so, and find that the calculation works out the same way, giving equation (1) for $n = 102$.

The pattern is clear, and the going becomes very quick: Copying the computation for $n = 102$, and just changing the exponents at the high end by 1, you get the case $n = 103$. Doing this again, you get the case $n = 104$; then $n = 105, \ldots$.

But is there any point in copying the calculation over and over? Surely you can do it once and for all. If at some step you have verified (1) for n equal to some value k, then in the next case you will have

(2) $$\begin{aligned} & 1/2 + 1/4 + \ldots + 1/2^{k+1} \\ = \ & (1/2 + 1/4 + \ldots + 1/2^k) + 1/2^{k+1} \\ = \ & (2^k - 1)/2^k + 1/2^{k+1} && \text{(by the case } n = k\text{)} \\ = \ & (2^{k+1} - 2)/2^{k+1} + 1/2^{k+1} \\ = \ & (2^{k+1} - 2 + 1)/2^{k+1} \\ = \ & (2^{k+1} - 1)/2^{k+1}. \end{aligned}$$

So there's no need to do any more calculations for particular values of n. Each later case indeed follows from the case before it, so equation (1) holds for *all* positive integers n.

2. Two more examples.

The kind of situation illustrated above comes up frequently. Let's note two more cases, before we abstract the general principle.

Many examples, like the one above, involve summation. For instance, suppose you try summing the first few *odd* integers:

$$\begin{aligned} 1 &= 1, \\ 1+3 &= 4, \\ 1+3+5 &= 9, \\ 1+3+5+7 &= 16. \end{aligned}$$

The sums are the first few squares, and these examples suggest the general formula

$$1 + 3 + \ldots + (2n-1) \ = \ n^2. \tag{3}$$

Is (3) true for all n? Again, if we have checked a given case $n = k$, then checking the next case reduces to a quick calculation:

$$\begin{aligned} & 1 + 3 + \ldots + (2(k+1)-1) \\ =\ & (1 + 3 + \ldots + (2k-1)) + (2(k+1)-1) \\ =\ & k^2 + (2k+1) \\ =\ & (k+1)^2. \end{aligned} \tag{4}$$

which is the $n = k+1$ case of (3). So since the $k = 1$ case is true, and each case implies the next, the result is true for all positive integers k.

For a slightly different sort of example, note that if we differentiate a function the form $x f(x)$, we get

$$(x f(x))' \ = \ f(x) + x f'(x).$$

If you differentiate this again, then differentiate the result, and so on (try a few steps!), you will see a pattern:

$$(x f(x))^{(n)} \ = \ n f^{(n-1)}(x) + x f^{(n)}(x). \tag{5}$$

To check whether that pattern will continue indefinitely, suppose the result true for n equal to some value k. Then one gets

$$\begin{aligned} & (x f(x))^{(k+1)} \\ =\ & ((x f(x))^{(k)})' \\ =\ & (k f^{(k-1)}(x) + x f^{(k)}(x))' && \text{(because we've assumed (5) for } n = k\text{)} \\ =\ & k f^{(k)}(x) + (f^{(k)}(x) + x f^{(k+1)}(x)) && \text{(using the law for differentiating a product)} \\ =\ & (k+1) f^{(k)}(x) + x f^{(k+1)}(x), \end{aligned} \tag{6}$$

which is the $n = k+1$ case of (5). So the pattern does indeed continue indefinitely.

3. The formal statement.

In each of the above examples, we dealt with an infinite family of statements, one for each positive integer n. (In the first case, these were the equations (1), in the second, the equations (3), and in the third, the equations (5).) In each case, we verified the first few of these statements, then did a general calculation that showed that the k-th statement implied the $(k+1)$-st; and we concluded from this that *all* of our statements held.

To get this conclusion, there was actually no need to check the first *few* cases − the very first case, together with the calculation showing that each case implied the next, would have been enough. (But the calculation of the first few cases still served an important purpose: It suggested *what* we should try to prove, without which we could not have set up the general calculation.)

To discuss this situation in the abstract, we need to give a name to the ''n-th statement'' of a pattern. Stewart, on p. 76, uses S_n; we will use the somewhat commoner notation $P(n)$, where P stands for ''proposition'', a logician's term for a statement. We can now formulate the

Principle of Mathematical Induction. *Suppose* $P(1)$, $P(2)$, $P(3)$, ... , $P(n)$, ... *are mathematical statements, and suppose we know that*

(i) $P(1)$ *is true,*

and that

(ii) *for all positive integers* k, $P(k)$ *implies* $P(k+1)$. (*I.e., if* $P(k)$ *is true, then* $P(k+1)$ *is true*).

Then $P(n)$ *is true for all* $n \geq 1$.

This principle can be used with various degrees of formality. In simple situations, one often uses it without calling it ''mathematical induction''. For instance, when one deduces from the formula for the derivative of the sum of *two* functions the corresponding law for the derivative of the sum of *any finite number* of functions, mathematical induction underlies the reasoning, but the reasoning is clear enough that it does not have to be spelled out. (So in that sense, you have been using mathematical induction for a long time.)

In less trivial situations, if one has a statement that one wants to prove for all n, one often shows that it is true for $n = 1$, shows that the n-th statement implies the $(n+1)$-st, and then says, ''Hence, by induction, the statement holds for all n''. In particular, it often happens that in the midst of a proof, one needs a statement which can be proved by induction. One may then say something like, ''We claim that for all $n \geq 1$, such-and-such is true. Indeed, it is true for $n=1$ because ... . Now assume inductively that it is true for $n = k$.'' One then shows why this implies that it is also true for $n = k+1$, and concludes, ''Hence, by induction, it is true for all n'', and goes on to use the fact in question. Note the words ''assume inductively'', which signal that Mathematical Induction is going to be used.

Finally, if the situation is complicated enough (or if one is learning the use of induction, and needs to show that one understands it), one may explicitly say, ''For each n, let $P(n)$ be the statement that ... ''. One then gives an argument showing that $P(1)$ is true, and an argument showing that $P(k)$ implies $P(k+1)$, and concludes, ''Hence, by Mathematical Induction, $P(n)$ is true for all n''.

4. Variants.

There are many slight variants to the version of the Principle of Mathematical Induction stated above.

The statement I gave started with $n = 1$. Clearly, the same reasoning would apply if we had a family of statements $P(n)$ starting with $n = 0$, and we proved that $P(0)$ held, and that each $P(k)$ implied $P(k+1)$: we could then conclude that $P(n)$ held for all $n \geq 0$. In some situations we might want to start with still another integer n_0, and our conclusion would be that $P(n)$ held for all $n \geq n_0$. (But 1 and 0 are the commonest cases.)

These slightly modified versions of induction can be proved *from* the version I gave. For instance, if we are given statements $P(n)$ for all $n \geq 0$ as above, we could define new statements $Q(1)$, $Q(2)$, ... by letting $Q(n)$ be $P(n-1)$. Then our original version of induction, applied to the statements $Q(n)$, yields the desired result about $P(0)$, $P(1)$, ...

Sometimes one may only have finitely many statements, say $P(1)$, $P(2)$, ... , $P(N)$ for some N. In that case, if one can prove $P(1)$, and show that $P(k)$ implies $P(k+1)$ for $1 \leq k < N$, then one can conclude that all of $P(1)$, $P(2)$, ... , $P(N)$ are true.

In another direction, the use of the distinct symbols n and k in the formulation of the Principle of Mathematical Induction is common, but not really necessary. It allows one to talk about ‘‘the case $n = k$’’ and ‘‘the case $n = k+1$’’; but one can equally well formulate condition (ii) of that statement as, ‘‘for all positive integers n, $P(n)$ implies $P(n+1)$’’, without switching to the letter k.

Note that the Principle of Mathematical Induction is specific to the integers. It is *not* true, for instance, that given a family of statements $P(x)$, one for each *real* number x, such that $P(1)$ is true and $P(x)$ implies $P(x+1)$ for all x, we can conclude that $P(x)$ holds for all real numbers ≥ 1. (For instance, the statement ‘‘x is an integer’’ has those properties, yet is not true for all real numbers $x \geq 1$.)

In a systematic development of the properties of the integers, certain axioms about their ordering and arithmetic are introduced, and the Principle of Mathematical Induction is deduced from these. (If you’d like to know more about this, ask me at office hours.) Some other sets, which satisfy some but not all of the properties of the integers, satisfy interesting variants of the Principle of Mathematical Induction.

One important variation on the method of Mathematical Induction, which again applies to the positive integers, and which can be proved from the usual form of Mathematical Induction, is called Complete Induction. Here we again deal with statements $P(1)$, $P(2)$, ... , $P(n)$, ... ; but rather than having to deduce $P(k+1)$ from $P(k)$ alone, as in condition (ii) above, we are allowed to use all of $P(1), ... , P(k)$ in proving $P(k+1)$. I won’t discuss it in detail here, but you can expect to see it in future math courses.

5. Circular reasoning?

A Teaching Assistant once described to me her experience teaching Mathematical Induction in first-year calculus. Berkeley was on the Quarter System then, so the courses were Math 1A-1B-1C. ‘‘In 1A, when you give an example of Mathematical Induction, some student is sure to say, ‘But teacher, you must be making a mistake. You’re assuming what you’re trying to prove!’ Then, in 1B, when you come to Mathematical Induction, someone will again say, ‘Something’s wrong. You’re assuming what you’re trying to prove.’ Finally, in 1C when you come to induction, they say, ‘Why are you spending time on this? We know it already!’ ’’

Well − when we ‘‘assume $P(k)$’’ in a proof by induction, are we, or aren’t we, assuming what we are trying to prove?

First note that we are not ‘‘assuming’’ $P(k)$ in the sense of taking for granted that it is true. Rather, we are saying, ‘‘If it is true ... ’’, and seeing what consequences that would have. Moreover, we are not arguing that $P(k)$ implies $P(k)$, which would be trivial, and the use of which to establish that all $P(n)$ hold would indeed be circular reasoning. Rather, we are showing that $P(k)$ implies $P(k+1)$. If this is so, and if P(1) is true, then we can correctly conclude by a ‘‘domino’’ effect that all $P(n)$ are true.

6. Errors to watch out for.

There are several errors I have seen among students learning to use Mathematical Induction.

The first is to check a result for $n = 1$, for $n = 2$, and maybe for a few more values, find that it is true in these cases, and say, ‘‘Therefore, by induction, it is true for all n’’. This is not valid unless one can come up with a precise argument showing why the truth of the statement for each value of n (not only those one has checked) implies its truth for the next value.

Second, a student will sometimes formulate a condition $P(k)$, prove $P(1)$, and then say, ‘‘By Mathematical Induction, $P(k)$ implies $P(k+1)$, hence $P(n)$ is true for all n’’. Such a student has gotten it backwards: Mathematical Induction does not tell us that $P(k)$ implies $P(k+1)$; rather, it says that *if*, using facts about the subject in question, we can show that $P(k)$ implies $P(k+1)$, and that $P(1)$ is true, *then* we can conclude that $P(n)$ holds for all n.

Students sometimes get the idea that any result that is to be proved for all n should be proved by induction.

Mathematical Induction is relevant in cases where there is a natural connection (other than analogy) between the statement for one value of n and the statement for the next. This is so for equations (1) and (3) above because the sum of $n+1$ terms is obtained from the sum of n terms by adding on the next term, and for (5) because the $(n+1)$-st derivative is obtained by differentiating the n-th derivative. An example where Mathematical Induction is not relevant is if one is asked to prove that for every positive integer n one has $(n^3)^3 = n^9$. The operation of cubing $n+1$ does not build on the operation of cubing n.

Here, finally, is a kind of error that does not come up often, but which makes an interesting brain-teaser. It is a well-known "proof" that all horses have the same color:

Let $P(n)$ be the statement that in every set of n horses, all the horses have the same color. Clearly $P(1)$ is true. Now suppose inductively that $P(k)$ is true, for some positive integer k. Given any set of $k+1$ horses, we can write it as the union of two sets of k horses, having $k-1$ horses in common. By our inductive assumption, all the horses in the first set have some common color, C_1, and all the horses in the second set have some common color C_2. But the $k-1$ horses that the two sets have in common are simultaneously of color C_1 and of color C_2; so $C_1 = C_2$; so our $k+1$ horses all have the same color, completing our inductive proof.

Can you find the fallacy?

7. Exercises.

Some exercises in Stewart relevant to Mathematical Induction are p. 76, exercise 20(a), §11.1, exercises 80-82 and 91 (pp. 705-706), and §11.2, exercise 91 (p. 714). Here are some more. In the exercises for §3.2, when I refer to the n-th derivative of a function f, this means f itself when $n = 0$.

3.2, "**Exercise 63**": Show that for every nonnegative integer n, the n-th derivative of e^{-x^2} can be written $p_n(x)\, e^{-x^2}$ where p_n is a polynomial of degree n.

3.2, "**Exercise 64**": Continuing the idea of the preceding exercise, suppose j is a positive integer. Prove an analogous result for the n-th derivative of e^{x^j}.

3.2, "**Exercise 65**": Suppose p and q are polynomials, and n a nonnegative integer. Prove that the n-th derivative of $p(x)/q(x)$ can be written as a rational function with denominator $q(x)^{n+1}$; i.e., that there is a polynomial $r_n(x)$ such that $d^n/dx^n\,(p(x)/q(x)) = r_n(x)/q(x)^{n+1}$.

§7.1, "**Exercise 73**": (a) Show that for every integer $n \geq 0$, one has $\int x^n e^x\, dx = p_n(x)\, e^x + C$ for some polynomial p_n of degree n. Give a formula by which p_n can be calculated from p_{n-1}, and give p_0.

The next part does not involve Mathematical Induction. It is simply a Chapter 7 exercise in obtaining the integral of one function from the integral of another.

(b) Deduce a result about $\int e^{x^{1/n}}\, dx$ for n a positive integer. (Hints: Use a substitution. The n in this part need not be the same as the n in the case of (a) which you will make use of.)

11.1, "**Exercise 96**": Let f_n denote the n-th Fibonacci number, as defined on p. 691. Prove that for all $n \geq 1$, $f_{n+1}\, f_{n-1} - f_n^2 = (-1)^n$.

Appendix E, "**Exercise 51**": Given a sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$, it may be hard to find a general formula for $\Sigma_{i=1}^{n} a_i$. However, given such a sequence, and a formula $\Sigma_{i=1}^{n} a_i = b_n$, I claim that it is easy to determine whether the formula is true. Namely,

(a) Given sequences of real numbers $a_1, a_2, \ldots, a_n, \ldots$ and $b_0, b_1, b_2, \ldots, b_n, \ldots$ with $b_0 = 0$, prove that the formula $\Sigma_{i=1}^{n} a_i = b_n$ holds for all positive integers n if and only if $b_n - b_{n-1} = a_n$ holds for all positive integers n.

(b) Using the above result, prove formula (e) on p. A37.