# Criteria for existence of semigroup homomorphisms and projective rank functions

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Suppose $A$, $S$, and $T$ are semigroups, $e: A \to S$ and $f: A \to T$ semigroup homomorphisms, and $X$ a generating set for $S$. We assume (1) that every element of $S$ divides some element of $e(A)$, (2) that $T$ is cancellative, (3) that $T$ is power-cancellative (i.e, $x^d = y^d \Rightarrow x = y$ for $d > 0$), and (4) a further technical condition, which in particular holds if $T$ admits a semigroup ordering with the order-type of the natural numbers. We show that there then exists a homomorphism $S \to T$ making a commuting triangle with $e$ and $f$ if and only if for every relation $w(x_1, \ldots, x_n) = e(a)$ holding in $S$, with $x_1, \ldots, x_n \in X$, $a \in A$, and $w$ a semigroup word, there exist $t_1, \ldots, t_n \in T$ satisfying $w(t_1, \ldots, t_n) = f(a)$.

This leads to an arithmetic criterion for the existence of integer-valued projective rank functions on rings.

## 1. Main results.

Semigroups will not be assumed to have neutral element. In the first sentence of the next definition, $S^1$ denotes the semigroup obtained by adjoining a neutral element to $S$.

**Definition 1.** *We shall say that an element* $s$ divides (*or is a* divisor *of*) *an element* $t$ *in a semigroup* $S$ *if* $t = psq$ *for some* $p, q \in S^1$.

*We shall say that an element* $s$ *is a* weak divisor *of a family* $G$ *of elements of* $S$ *if there exists some positive integer* $m$ *such that* $s^m$ *is a divisor of an element of* $G^m$ (*i.e., of a product* $g_1 \ldots g_m$ *with* $g_i \in G$).

The only excuse for this peculiar definition, and for condition (4) of the next theorem, where it is used, is that this is what was needed to abstract an argument discovered in the case where $T$ was the additive semigroup of natural numbers. (Note that if $a$ is a weak divisor of $\{b\}$ and $b$ a weak divisor of $\{c\}$, $a$ need not be a weak divisor of $\{c\}$.) In §2 we will look at some simpler hypotheses that imply (4); in the mean time, the reader may take for granted that (4) holds frequently, e.g., whenever $T$ is a free semigroup or a free abelian semigroup. Experience may eventually show that one of the simpler hypotheses to be mentioned covers all cases of interest.

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**Theorem 2.** *Let* $A$, $S$, *and* $T$ *be semigroups,* $e: A \to S$ *and* $f: A \to T$ *semigroup homomorphisms, and* $X$ *a generating set for* $S$. *Suppose that*

(1) *every element of* $S$ *divides some element of* $e(A)$,

(2) $T$ *is right and left cancellative* (*i.e.,* $axb = ayb \Rightarrow x = y$),

(3) $T$ *is power cancellative,* ($x^d = y^d \Rightarrow x = y$ *for* $d > 0$),

*and*

(4) *every finite subset of* $f(A) \subseteq T$ *has only finitely many weak divisors, and each of these in turn has only finitely many divisors.*

*Then there exists a homomorphism* $S \to T$ *making a commuting triangle with the given maps from* $A$ *if and only if for every relation* $w(x_1, \ldots, x_n) = e(a)$ *holding in* $S$, *with* $x_1, \ldots, x_n \in X$, $a \in A$, *and* $w$ *a semigroup word, there exist* $t_1, \ldots, t_n \in T$ *satisfying* $w(t_1, \ldots, t_n) = f(a)$.

**Proof.** It is clear that a homomorphism $S \to T$ is determined by its restriction to $X$, say $\tau: X \to T$, and that it will form a commuting triangle with $A$ if and only if $\tau$ has the property that for every relation

(5) $$w(x_1, \ldots, x_n) = e(a)$$

holding in $S$ with $x_1, \ldots, x_n \in X$, $a \in A$, one has

(6) $$w(\tau(x_1), \ldots, \tau(x_n)) = f(a)$$

in $T$. We claim that given an arbitrary set-map $\tau: X \to T$, the condition

(7) $$(\forall\, w, x_1, \ldots, x_n, a)\ (5) \Rightarrow (6)$$

in fact implies that $\tau$ is the restriction of such a homomorphism. To get the latter condition, it clearly suffices to show that given any relation

(8) $$w_1(x_1, \ldots, x_m) = w_2(x_1, \ldots, x_m)$$

holding in $S$, the corresponding relation holds between the $\tau(x_i)$ in $T$. Now writing $s$ for the common value of the two sides of (8), hypothesis (1) says that $s$ divides $e(a)$ for some $a \in A$. Expressing the right and left factors that carry $s$ to $e(a)$ in terms of the generating set $X$ (using a list of elements of $X$ extending $x_1, \ldots, x_n$), we get

$$u(x_1, \ldots, x_n)\, w_1(x_1, \ldots, x_m)\, v(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)\, w_2(x_1, \ldots, x_m)\, v(x_1, \ldots, x_n) = e(a).$$

Regarding this as two equations of the form (5), we see that (7) implies that the corresponding equations (6) hold in $T$, so by the cancellativity of $T$, the equation corresponding to (8) also holds in $T$, as desired.

Let $\Sigma_0$ denote the system of semigroup equations (6) in an $X$-tuple of $T$-valued unknowns $(\tau(x))_{x \in X}$ obtained by replacing the elements $x \in X$ in the left-hand sides of all equations (5) holding in $S$ with the corresponding elements $\tau(x)$, and the elements $e(a)$ on the right-hand-sides by the elements $f(a) \in T$. What we have shown is that homomorphisms $S \to T$ making the indicated triangle commute correspond to solutions in $T$ to the system $\Sigma_0$. We shall now assume that there exists no such homomorphism, i.e., that $\Sigma_0$ has no solution, show that $\Sigma_0$ must then have a finite subset with no solution, and finally construct a single member of $\Sigma_0$ with no solution. This will show the ''if'' direction of the Theorem; ''only if'' is clear.

We begin by noting that by condition (1), each $x \in X$ is a divisor of some element $e(a_x)$, hence there is some member of $\Sigma_0$, say $U_x = f(a_x)$, such that the word $U_x$ involves the unknown $\tau(x)$. For each $x$, let us fix one such element of $\Sigma_0$, and denote by $T_x$ the set of divisors in $T$ of the element $f(a_x)$. We see from (4) that each $T_x$ is finite. (We will not use the full strength of (4) till later.) Note that in any solution in the semigroup $T$ of any subsystem of $\Sigma_0$ which includes the equation $U_x = f(a_x)$, the value given to $\tau(x)$ must be a member of the finite set $T_x$.

Let us now take the product set $T_X = \Pi_X\, T_x$, and regard it as a compact topological space under the product of the discrete topologies on the finite factors. The set of solutions in $T_X$ to each equation in $\Sigma_0$ is clearly closed. We have assumed that the intersection over $\Sigma_0$ of these closed sets is empty, hence by compactness, the intersection over some finite subset $\Sigma_1 \subseteq \Sigma_0$ is empty. Let $\tau(x_1), \dots, \tau(x_m)$ be the unknowns occurring in the equations of $\Sigma_1$. Thus no choices of values for $\tau(x_1), \dots, \tau(x_m)$ in the respective sets $T_{x_1}, \dots, T_{x_m}$ will satisfy $\Sigma_1$. Now let $\Sigma_2$ be obtained by adjoining to $\Sigma_1$ the equations $U_{x_i} = f(a_i)$ for $i = 1, \dots, m$. Any solution to the latter equations *must* have $\tau(x_1), \dots, \tau(x_m)$ in $T_{x_1}, \dots, T_{x_m}$ respectively. Thus $\Sigma_2$ is a finite subsystem of $\Sigma_0$ having no solution.

In passing from $\Sigma_1$ to $\Sigma_2$ we may have expanded the set of variables occurring; let us write the resulting list $\tau(x_1), \dots, \tau(x_n)$. We now claim that we can modify $\Sigma_2$ to get a system $\Sigma_3$ in this same set of variables, also with no solution in $T$, having the additional property that all $n$ variables occur in every equation. To do this, we go successively through the list of $n$ variables; for each variable $\tau(x_i)$, we find one equation $U = f(a)$ in our system which *does* involve it, and if some other equations do not, we multiply each of these on the left by $U = f(a)$. Since we have kept the latter equation in our system, and since $T$ is cancellative, it is easy to see that a solution to our system after the above modification will be the same as a solution to the system before modification. Going through all the variables this way, we get the desired $\Sigma_3$.

Let us list the equations comprising $\Sigma_3$ as $U_j = b_j$ ($j = 1, \dots, r$, $b_j \in f(A)$). We shall now construct from these a single equation

$$(9) \qquad (U_1)^{d_1} \dots (U_{r-1})^{d_{r-1}}\, U_r = (b_1)^{d_1} \dots (b_{r-1})^{d_{r-1}}\, b_r \qquad \text{with } d_1, \dots, d_{r-1} \geq 0$$

having no solution in $T$. To do this, we must again begin by finding a finite set which limits the possible values that can be assigned to $\tau(x_1), \dots, \tau(x_n)$. Let $B$ denote the set of all products of length $\leq r$ of elements of $\{b_1, \dots, b_r\}$, and let $T_0 \subseteq T$ be the set of all *divisors* of *weak divisors* of the finite family $B$. This is a finite set by hypothesis (4). Now given any equation of the form (9), if we let $d_j > 0$ be maximal among the $d$'s (where we understand $d_r = 1$), then we see that the right-hand side of (9) may be written (in one way or another) as a product of $d_j$ elements of $B$, and the left-hand-side has $(U_j)^{d_j}$ as a divisor. Hence in any solution to this equation in $T$, the value assumed by $U_j$ will be a weak divisor of the set $B$, and the value assumed by every $\tau(x_i)$ will be a divisor of this value, hence will belong to $T_0$.

Any equation of the form (9) belongs to $\Sigma_0$, hence to get an equation in $\Sigma_0$ having no solution in $T^X$, it suffices to choose (9) having no solution in $(T_0)^n$.

By induction on $r$, we can do this if we can show that we can obtain from the last two equations of the system $\Sigma_3$,

$$(10) \qquad U_{r-1} = b_{r-1},$$

$$(11) \qquad U_r = b_r,$$

a single equation

(12) $$(U_{r-1})^d U_r = (b_{r-1})^d b_r$$

whose solution-set in $(T_0)^n$ is precisely the intersection of the solution-sets of (10) and (11). For this, in turn, it will certainly suffice to show that if a given family of values of $\tau(x_1), \dots, \tau(x_n)$ in $T_0$ is not a solution to *both* (10) and (11), then there exists at most one value of $d$ for which this family *is* a solution to (12). Now given an element of $(T_0)^n$ at which (10) but not (11) holds, we see that (12) will not hold for any $d$, by (2). On the other hand, if (10) does not hold, suppose (12) holds for some value of $d$, and let $d_0$ be the least such value. For any higher value $d_1$, we can write the $d_1$-case of (12) as the product of the $d_0$-case thereof, which holds, and the $(d_1 - d_0)$th power of (10), which by (3) does not hold; hence this case of (12) fails by (2). We conclude that for all but at most $\mathrm{card}(T_0)^n$ values of $d$, and so in particular, for at least one value, the solution set of (12) is the intersection of the solution-sets of (10) and (11), as claimed. The indicated induction now leads to an equation (9) having empty solution-set, completing the proof of the Theorem. □

The original motivation for the above Theorem came from the study of integer-valued projective rank functions on rings; cf. [**2**]. Let us apply it to that case.

**Corollary 3.** *Let $R$ be an associative ring, and $X$ a set of finitely generated nonzero projective left $R$-modules, such that every finitely generated projective left $R$-module is isomorphic to a direct sum of members of $X$.*

*Then there exists a* projective rank function *for $R$ (a function from isomorphism classes of finitely generated projective left $R$-modules to natural numbers carrying direct sums of modules to sums of integers, and $R$ to* 1) *if and only if whenever one has an isomorphism of modules*

(13) $$(P_1)^{c_1} \oplus \dots \oplus (P_n)^{c_n} \cong R^c \qquad (P_1, \dots, P_n \in X,\ c, c_1, \dots, c_n \geq 0),$$

*the integer $c$ is a linear combination of $c_1, \dots, c_n$, with nonnegative integer coefficients.*

*This rank function can be taken to be* faithful (*to carry nonzero projective modules to positive integers*) *if and only if whenever* (13) *holds, the integer $c$ is a linear combination of $c_1, \dots, c_n$ with* positive *integer coefficients.*

**Proof.** To get the first assertion, apply the preceding Theorem taking for $S$ the semigroup of isomorphism classes of nonzero finitely generated projective $R$-modules (and mumbling the words needed to replace these proper classes by genuine sets), for both $A$ and $T$ the additive semigroup $\mathbf{N}$ of the nonnegative integers, for $e$ the map taking 1 to the isomorphism class of the free $R$-module of rank 1, and for $f$ the identity map. Condition (1) of the Theorem holds because every finitely generated projective module is a direct summand in a free module of finite rank; (2) and (3) are obvious, and we see (4) by noting that in the additive semigroup of nonnegative integers, an element is a ‘‘divisor’’ of another if and only if it is majorized by that element under the natural ordering of the integers, and is a ‘‘weak divisor’’ of a family if and only if it is majorized by some member of that family; and every nonnegative integer majorizes only finitely many others. The criterion of Theorem 2 now assumes the desired form (the coefficients of this Corollary corresponding to the $t_i$ of the Theorem).

The final assertion is obtained by the same argument, using the semigroup of positive integers in place of the nonnegative integers. □

In [**3**], generalized projective rank functions, with values in semigroups $(1/n)\mathbf{N}$, are used to study homomorphisms of $R$ into $n \times n$ matrix rings over division rings. The same method as above shows that $R$ admits such a function if and only if for every equation (13), the integer $nc$ can be written as a linear combination of the $c_i$ with nonnegative (respectively, positive) integer coefficients.

In our proof of Theorem 2, we first showed that the infinite system of equations $\Sigma_0$ could be replaced by a finite system $\Sigma_2$ without using the full strength of our hypotheses. Hence let us record

**Corollary 4** (to proof of Theorem 2)**.** *Assume the hypothesis of Theorem 2, but without condition* (3) (*power cancellativity*)*, and with* (4) *weakened to say merely that every element of* $f(A)$ *has only a finite number of divisors in* $T$*. Then a necessary and sufficient condition for there to exist a homomorphism* $S \to T$ *making a commuting triangle with the given maps from* $A$ *is that for every* finite family *of relations* (5) *in a common set of elements of* $x_1, \ldots, x_n \in X$ *holding in* $S$*, there exist* $t_1, \ldots, t_n \in T$ *satisfying the corresponding equations* (6) *in* $T$*.* □

## 2. Implications and examples.

Condition (4) of our Theorem is messy. What simpler conditions can we give under which this hypothesis will hold?

A statement which clearly implies (4), and is simpler both in that it does not involve divisors of divisors, and in that it refers only to $T$, and not to $f(A)$, is

(4′) Every finite subset of $T$ has only finitely many weak divisors.

(This is equivalent to (4) holding when $A = T$ and $f$ is the identity map. Incidentally, a change that would lead to a slight simplification in the formulation of (4) itself, would be to redefine ‘‘weak divisor’’ to mean what we are calling ‘‘divisor of a weak divisor’’ – the latter concept may be more natural than the one we have defined.)

A much stronger condition, which implies all of (2), (3) and (4′), is

(14) $T$ admits a total ordering with the property
$$a < b \Rightarrow ca < cb \text{ and } ad < bd \text{ for all } a, b, c, d \in T,$$
and having the order-type of the natural numbers.

Indeed, the argument by which we showed in the proof of Corollary 4 that (4) held if $T$ was the semigroup of nonnegative or of positive integers works more generally under the assumption (14), once we verify that the conditions on the ordering in (14) imply a couple of other properties that were obvious in those cases. First, if $T$ has an idempotent $e$, this must be a neutral element. Indeed, given an idempotent $\mathbf{e}$, if for any $a$ the product $ea$ were $< a$ or $> a$, then from (14) we would get the same inequality between $e^2 a$ and $ea$, contradicting the idempotence of $e$; so $e$ is a left neutral element, and by the symmetric argument it is a right neutral element. Secondly, for all $a, b$ with $a$ not a neutral element, we have $ab > b$. For if we had $b > ab$ or $b = ab$, we would get $ab > a^2 b$, respectively $ab = a^2 b$. In the first case, we could go on to get an infinite descending chain $b > ab > a^2 b > a^3 b \ldots$, contradicting our order-type hypothesis, while in the second we would get $a^2 = a$, so by our previous observation, $a$ would be a neutral element. Again, we similarly have $ba > b$. Given the latter property, the argument cited offers no difficulty.

Easy examples of such ordered semigroups are given by the subsemigroups of the additive group of real numbers generated by increasing sequences of positive numbers having no upper bound, for example $\{2^1/_2, 3^1/_3, \ldots, n + n^{-1}, \ldots\}$. If we take linearly independent generators, such as the powers of $\pi$, the resulting semigroup is free abelian; thus, free abelian semigroups on countably many generators satisfy (14). The same is true of free nonabelian semigroups; the interested reader can easily find the appropriate orderings, but instead of establishing (2)-(4′) in this way, let us note that (2) and (3) clearly hold, and give another general principle for establishing (4) and (4′):

(15) If a semigroup $T$ admits a finitely-many-to-one homomorphism to a semigroup satisfying (4′), or satisfying (4) with respect to a subsemigroup $f(A)$, then $T$ itself also satisfies (4′), respectively satisfies (4) with respect to the image of $f(A)$ under that homomorphism.

This is immediate. Now the free semigroup on generators $x_1$, $x_2$, ... can be mapped to the free semigroup on one generator $x$ (which we have seen satisfies (4′)) by sending $x_i$ to $x^i$, and this map is finitely-many-to-one, hence this free semigroup satisfies (4′). The same argument proves that (4) holds for free semigroups in any semigroup variety not satisfying an identity $(\forall x)(x^m = x^n)$. (Some of these varieties have, and some do not have the property that their free semigroups also satisfy (2) and (3).)

The above methods are not directly helpful for *uncountable* semigroups. However, it is not hard to see that a semigroup satisfies (4′) if and only if all of its countable subsemigroups do; hence, *all* free semigroups satisfy (4′).

An example showing that the order-type condition in (4′) cannot be weakened to ‘‘well-ordered’’ is given by the additive semigroup $T$ generated by $\{1/2,\ 2/3,\ 3/4,\ \ldots,\ n/n{+}1,\ \ldots\}$. Since $T$ is generated by a set well-ordered under the usual ordering of the real numbers, it is itself well-ordered under this ordering, [**1**, Theorem III.2.9, p. 123]. But if $p, q$ are members of this semigroup with $p \leq q$, it is not hard to show that $p$ is a weak divisor of $\{q\}$. (For the $m$ in the definition of weak divisor, use a common denominator of $p$ and $q$, and remember that since $T$ contains $1/2$, it contains all positive integers.) Hence, the singleton set $\{1\}$ has infinitely many weak divisors in this semigroup. I do not know whether there is a counterexample to the conclusion of Theorem 2 for this $T$.

Let us note record another variant of our Theorem, easily obtained by the same proof

**Corollary 5** (to proof of Theorem 2)**.** *Suppose in the situation of Theorem 2 that for each* $x \in X$ *we are given a subset* $Y_x \subseteq T$, *and that condition* (4) *is weakened to say that the set of divisors of weak divisors of any finite subset of* $f(A)$ *has finite intersection with each* $A_x$. *Then the conclusion still holds, if the homomorphisms* $S \to T$ *referred to are required to carry* $x$ *into* $Y_x$ *for each* $x \in X$, *and the solutions to* $w(t_1, \ldots, t_n) = e(a)$ *are likewise required to satisfy* $t_i \in Y_{x_i}$ $(i = 1, \ldots, n)$. □

We end with an example where all hypotheses of Theorem 2 are satisfied except (3), but the conclusion of the Theorem fails. Let $S$ be the subsemigroup of the group $(Z_2)^3 \times Z$ generated by the set $X$ consisting of the seven elements $(\alpha, 1)$ with $\alpha \in (Z_2)^3 - \{0\}$. Let $T$ be the subsemigroup of $(Z_2)^2 \times Z$ similarly generated by the set of three elements $(\beta, 1)$ $(\beta \in (Z_2)^2 - \{0\})$, which we shall call $Y$. Let $A$ be the semigroup of positive integers $> 1$, and map $A$ into both $S$ and $T$ by $n \mapsto (0, n)$. Conditions (1) and (2) are clear, and (4) holds because $T$ has an obvious finitely-many-to-one homomorphism to the positive integers. We also claim that every relation (5) holding in $S$ yields an equation (6) satisfiable in $T$: Assuming (5), we see that either no element of $X$, or more than one elements of $X$ appear with *odd* exponent-sum on the left hand side of that equation. Let us map the set of elements of $X$ with this property to a set of elements of $Y$ whose first components sum to $0$, and map the remaining elements of $X$ to arbitrary elements of $Y$; we find that this assignment satisfies (6). However, it is easy to verify that a homomorphism $S \to T$ making a commuting triangle with $A$ would induce a homomorphism $(Z_2)^3 \to (Z_2)^2$ carrying no nonzero element to $0$, which is impossible.

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