Comments, corrections, and related references welcomed, as always!

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SOME RESULTS ON COUNTING LINEARIZATIONS OF POSETS

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Abstract. In section 1 we consider a 3-tuple $S = (|S|, \preccurlyeq, E)$ where $|S|$ is a finite set, $\preccurlyeq$ a partial ordering on $|S|$, and E a set of unordered pairs of distinct members of $|S|$, and study, as a function of $n \geq 0$, the number of maps $\varphi : |S| \to \{1, \ldots, n\}$ which are both isotone with respect to the ordering $\preccurlyeq$, and have the property that $\varphi(x) \neq \varphi(y)$ whenever $\{x, y\} \in E$. We prove a number-theoretic result about this function, and use it in section 8 to recover a ring-theoretic identity of G. P. Hochschild.

In section 2 we generalize a result of R. Stanley on the sign-imbalance of posets in which the lengths of all maximal chains have the same parity.

In sections 3-6 we study the linearization-count and sign-imbalance of a lexicographic sum of n finite posets P_i $(1 \leq i \leq n)$ over an n-element poset P_0. We note how to compute these values from the corresponding counts for the given posets P_i, and for a lexicographic sum over P_0 of chains of lengths $\mathrm{card}(P_i)$. This makes the behavior of lexicographic sums of chains over a finite poset P_0 of interest, and we obtain some general results on the linearization-count and sign-imbalance of these objects.

This material is far from my areas of expertise. The referees for two journals were not enthusiastic about it, so I have decided not to publish it, but I am keeping it available online in case it should prove of interest to someone.

1. Denominators of order-chromatic polynomials

In this section, for $S = (|S|, \preccurlyeq, E)$ as in the first paragraph of the abstract, we show that the function of n defined in that sentence is a polynomial with rational coefficients, and we obtain a bound on the primes dividing its denominator. This bound is used, as indicated, in section 8 (an appendix, which depends only on the present section).

Some background: Recall that the *chromatic polynomial* of a finite graph G is the function f associating to every positive integer n the number of colorings of the vertices of G with n colors (not all of which need be used), such that adjacent vertices have different colors. (We will recover below the fact that f is indeed a polynomial function.)

Richard Stanley notes in [4] that if $P = (|P|, \preccurlyeq)$ is a finite poset (partially ordered set), then the function e associating to every positive integer n the number of isotone maps $\varphi : P \to \{1, \ldots, n\}$ (i.e., maps $\varphi : |P| \to \{1, \ldots, n\}$ such that $x \preccurlyeq y \implies \varphi(x) \leq \varphi(y)$) is also given by a polynomial, conceptually similar to the chromatic polynomial of a graph, and that the same is true of the function $\overline{e}$ associating to n the number of *strictly* isotone maps $P \to \{1, \ldots, n\}$ (maps such that for distinct x, y, one has $x \preccurlyeq y \implies \varphi(x) < \varphi(y)$).

To see the above fact for the function e, let us break the process of choosing an isotone map $\varphi : P \to \{1, \ldots, n\}$ into three steps. First, decide which elements will fall together under φ. Calling the quotient poset arising from these identifications P', choose, next, the linear order which the embedded image of $|P'|$ in $\{1, \ldots, n\}$ is to have. Finally, choose a way of embedding the resulting linearly ordered set into $\{1, \ldots, n\}$. Now if we write

$$(1) \qquad m = \mathrm{card}(P),$$

then the number of elements of each P' will be some $m' \leq m$, and once a linear ordering of such a $|P'|$ has been chosen, the number of embeddings of the resulting linearly ordered set in $\{1, \ldots, n\}$ will be the

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number of ways of choosing m' out of n elements, which is the binomial coefficient $\binom{n}{m'} = n(n-1)\dots(n-m'+1)/m'!$. This is a polynomial in n of degree m' over the rational numbers. Summing the polynomials arising in this way from all linearly ordered sets obtained as in the first two steps above, we get a polynomial of degree m.

Now the coefficients of this polynomial can clearly be written to the common denominator $m!$, so every prime dividing the least common denominator of those coefficients is $\leq m$. But we can get a better bound on such primes if the Hasse diagram of P has more than one connected component.

To do so, let us modify the procedure described above. The first step, choosing which elements of $|P|$ fall together, remains the same; again let P' denote the set so obtained, given with the weakest partial ordering making the map $P \to P'$ isotone. (Here we are assuming that $|P'|$ is a set-theoretic image of $|P|$ which admits a partial ordering making the map $|P| \to |P'|$ isotone. The weakest such partial ordering is then the intersection of all such partial orderings.) At the second step, rather than strengthening that ordering to a linear ordering on $|P'|$, let us strengthen it to an ordering which is linear on each connected component of the Hasse diagram of P', but leaves elements of distinct connected components incomparable. Finally, we choose a one-to-one isotone map of the resulting poset into $\{1,\dots,n\}$.

How many choices are there at that last step? If, under the ordering chosen at the second step, the connected components of P' are chains $P'_1,\dots,P'_k$, of cardinalities $m'_1,\dots,m'_k$ respectively, then there are $\binom{n}{m'_1}$ isotone embeddings of P'_1 in $\{1,\dots,n\}$, then $\binom{n-m'_1}{m'_2}$ such embeddings of P'_2 in the remaining elements of $\{1,\dots,n\}$, and so on. So the total number of isotone embeddings is $\binom{n}{m'_1}\binom{n-m'_1}{m'_2}\dots\binom{n-m'_1-\dots-m'_{k-1}}{m'_k}$, which, writing $m'_1+\dots+m'_k = m' = \mathrm{card}(|P'|)$, is the multinomial coefficient

$$n(n-1)\dots(n-m'+1)/m_1!\dots m_k!\,. \tag{2}$$

This is a polynomial in n whose denominator is divisible only by primes that are less than or equal to one of the m'_i, so we need to know how large the m'_i, the cardinalities of the components of P', can be.

Suppose the original poset P had c connected components. Each identification of two elements of P reduces the number of connected components by at most one, hence if P' has m' elements, and so has undergone $m-m'$ identifications, it still has at least $c-(m-m')$ components. The largest that such a component can be is m' minus the number of *other* components, so each component has at most $m'-(c-(m-m')-1) = m-c+1$ elements.

Thus, $m-c+1$ is an upper bound on the primes that can occur in the denominator of (2). Summing over all choices of our image P' of P, and all componentwise linearizations thereof, we conclude that the polynomial giving the number of isotone maps $P \to \{1,\dots,n\}$ has denominator divisible only by primes less than or equal to $m-c+1$.

As noted in the abstract, we actually want to count the smaller set of such maps $\varphi : P \to \{1,\dots,n\}$ that satisfy an additional set of restrictions on which pairs of elements of P can fall together under φ. But such restrictions do not affect the above argument: they limit the set of images P' that we enumerate at the first step, while our bound on primes in the denominator comes from the last step. So we have

Theorem 1. *Let S be a 3-tuple $(|S|,\preccurlyeq,E)$, where $|S|$ is a finite set, $\preccurlyeq$ a partial ordering on $|S|$, and E a set of unordered pairs of distinct members of $|S|$. Let c be the number of connected components of the Hasse diagram of the poset $(|S|,\preccurlyeq)$, and for each $n \geq 0$, let $C(S,n)$ be the number of maps $\varphi : |S| \to \{1,\dots,n\}$ which are both isotone with respect to the ordering $\preccurlyeq$, and have the property that $\varphi(x) \neq \varphi(y)$ whenever $\{x,y\} \in E$.*

Then $C(S,n)$ is a polynomial in n with rational coefficients, such that all primes dividing the denominators of those coefficients are $\leq \mathrm{card}(|S|)-c+1$. □

Note that in the case of the above result where $\preccurlyeq$ is the trivial ordering, i.e., where S is an antichain, $C(S,n)$ is the chromatic polynomial of the graph $(|S|,E)$. In that case $c = \mathrm{card}(|S|)$, so the theorem says $C(S,n)$ is a polynomial with *integer* coefficients; which is indeed true of chromatic polynomials [7, §IX.2].

Theorem 1 actually remains true if we allow arbitrary Boolean conditions on which elements fall together. For instance, for particular $x_1,\ x_2,\ y_1,\ y_2 \in |S|$, the condition, "if x_1 falls together with y_1, then x_2 falls together with y_2" would cause no difficulties with the proof. I have assumed the conditions to have the simple form stated in the theorem because such conditions seem to occur naturally; in particular, the problem that motivated this result, studied in section 8 below, has conditions of that form. The result for arbitrary

Boolean expressions in such conditions is, in any case, easily deduced from Theorem 1 by inclusion-exclusion considerations.

On the other hand, Theorem 1 does not remain true if we allow our additional restrictions to involve the *order* relations among elements $\varphi(x)$; for instance, if for some particular $x, y \in |S|$ we consider only maps φ such that $\varphi(x) \leq \varphi(y)$. To impose such a condition is equivalent to replacing the P of our introductory discussion by a poset with its order relation $\preccurlyeq$ strengthened; and this can fuse distinct connected components without reducing the number of elements. So, for instance, if $(|S|, \preccurlyeq)$ is an m-element antichain, then an obvious set of conditions of the above form restricts us to maps $|S| \to \{1, \ldots, n\}$ which are isotone with respect to a particular total ordering $\preccurlyeq'$ of $|S|$, and the number of such maps is $\binom{n+m-1}{m}$, a polynomial with denominator $m!$, despite the fact that the original antichain ordering $\preccurlyeq$ satisfies $m - c + 1 = 1$.

The following consequence of Theorem 1 will be used in section 8. Recall that the chromatic number of a graph is the least positive integer n at which the chromatic polynomial of the graph is nonzero.

Corollary 2. *Let $S = (|S|, \preccurlyeq, E)$ and c be as in Theorem 1, and p be any prime greater than* $\mathrm{card}(|S|) - c + 1$.

Then for every positive integer n whose residue modulo p is less than the chromatic number of the graph $(|S|, E)$, the integer $C(S, n)$ is divisible by p. In particular, if $|S|$ is nonempty, then $C(S, n)$ is divisible by p whenever n is divisible by p.

Proof. By Theorem 1 we may write $C(S, n) = f(n)/r$, where f is a polynomial with integer coefficients, and r an integer not divisible by p. The residue of $f(n)$ modulo p depends only on the residue of n modulo p, hence as r is invertible modulo p, the same is true of the residue of $C(S, n) = f(n)/r$. Now for positive n_0 less than the chromatic number of $(|S|, E)$, there are no maps of $|S|$ into $\{1, \ldots, n_0\}$ which distinguish x and y whenever $\{x, y\} \in E$, in particular, no isotone maps with that property; hence $C(S, n_0) = 0$ for such n_0. Hence for n congruent to a value n_0 in that range, we have $C(S, n) \equiv_{(\mathrm{mod}\ p)} C(S, n_0) = 0$.

The final sentence follows because every nonempty graph has chromatic number greater than zero. □

The remaining sections of this paper, other than the appendix section 8, are related to this one only in that they involve counting linear images of posets, and use somewhat similar methods.

A belated remark on notation: At the end of the third paragraph of this section, I spoke of the condition that "for distinct x, y, one has $x \preccurlyeq y \implies \varphi(x) < \varphi(y)$", where it would be more natural to say that for all x, y, one has $x \prec y \implies \varphi(x) < \varphi(y)$. But I am avoiding the use of "$\prec$" for strict $\preccurlyeq$-inequality because many authors use $x \prec y$ for the condition that y *covers* x with respect to a partial order $\leq$, i.e., that $x < y$ and there is no element z with $x < z < y$. At a few places below, we will indeed consider the covering relation, though we will not introduce any notation for it. However, I felt it best to avoid confusion with a notation commonly used by others.

2. The sign-imbalance of a bicolorable poset

If X is a set of $n < \infty$ elements, then there are precisely $n!$ bijections $X \to \{1, \ldots, n\}$, and given one such bijection b, we can write any other as $\pi\, b$ for a unique permutation π of $\{1, \ldots, n\}$. In this situation, one calls a bijection $X \to \{1, \ldots, n\}$ "even" or "odd" (relative to b), or equivalently, of sign $+1$ or -1, according to whether π is an even or an odd permutation. Clearly, replacing b with another bijection $X \to \{1, \ldots, n\}$ either preserves or reverses this classification of bijections.

If $P = (|P|, \preccurlyeq)$ is an n-element poset, then the *isotone* bijections $|P| \to \{1, \ldots, n\}$ correspond naturally to linearizations of P (total orderings refining $\preccurlyeq$), and one calls a linearization even or odd if the corresponding bijection is so. The number of linearizations that are even minus the number that are odd is called the *sign-imbalance* of the poset P (see [5] and papers referenced there). If that number is zero, P is called *sign-balanced*. The sign-imbalance is, of course, unique only up to sign, unless a particular ordering b relative to which it is calculated has been specified.

In studying the sign-imbalance of a finite poset P, our method will again be to break the choice of a linearization of P into steps, such that for each partial ordering achieved at the next-to-last step, the number of choices available at the final step is easy to describe.

Since only one-to-one images of P are considered, we have no use for a graph structure E on $|P|$ restricting which elements fall together. We will, however, make use of the graph structure of the Hasse diagram of P, via

Definition 3. *A* bicoloring *of a finite poset* $P = (|P|, \preccurlyeq)$ *will mean a coloring of its Hasse graph with two colors; that is, a partition of* $|P|$ *into two subsets, such that whenever one element of* $|P|$ *covers another, the elements belong to different members of the partition (“have opposite colors”).*

Thus, P admits a bicoloring if and only if every cycle in its Hasse diagram has an even number of vertices.

If $P = (|P|, \preccurlyeq)$ is a finite poset on which we have a bicoloring, and $\leq$ is some total ordering refining the given partial ordering $\preccurlyeq$, let us break $|P|$ into the “blocks” of elements of the same color that are consecutive under $\leq$. That is, let $|P|_1$ be the subset of $|P|$ consisting of the least element under $\leq$, and all elements of the same color (if any) that follow it under $\leq$ without an intermediate element of the opposite color; let $|P|_2$ consist of the first element of the other color, and those that similarly follow it, $|P|_3$ the next block (which has the same color as $|P|_1$), and so on. Note that within each $|P|_i$, all elements are incomparable under $\preccurlyeq$, since by the definition of a bicoloring, any two comparable elements of the same color have an element of the opposite color between them under $\preccurlyeq$, and hence under $\leq$.

In the above situation, we can define an intermediate partial ordering $\leq_{\mathrm{pre}}$ on $|P|$, under which members of each $|P|_i$ are incomparable, as they are under $\preccurlyeq$, while whenever $x \in |P|_i$ and $y \in |P|_j$ with $i < j$, we let $x \leq_{\mathrm{pre}} y$. We shall call the partial ordering $\leq_{\mathrm{pre}}$ the “prelinearization” of $\preccurlyeq$ determined by the linearization $\leq$.

Let us call linearizations $\leq$ and $\leq'$ of $\preccurlyeq$ “block-equivalent” if the prelinearizations $\leq_{\mathrm{pre}}$ and $\leq'_{\mathrm{pre}}$ are the same. Given a prelinearization of P, with block decomposition $|P| = |P|_1 \cup |P|_2 \cup \cdots \cup |P|_m$, the set of linearizations $\leq$ of $\preccurlyeq$ that determine that prelinearization (a block-equivalence-class of linearizations of $\preccurlyeq$) will have exactly $\mathrm{card}(|P|_1)!\ \mathrm{card}(|P|_2)! \ldots \mathrm{card}(|P|_m)!$ elements, every such linearization being obtained by choosing an arbitrary linearization of each $|P|_i$. The sign of each such linearization of P will be, up to a fixed factor ± 1, the product of the signs of the linearizations of these subsets.

Now if any $|P|_i$ has cardinality greater than 1, then the $\mathrm{card}(|P|_i)!$ choices for the ordering of that block will be half even and half odd, so the parities of the linearizations in the whole block-equivalence class will be equally split between odd and even. Hence in computing the summation of $+1$'s and -1's that gives the sign-imbalance of P, it suffices to look at linearizations in which each same-color block is a singleton; in other words, linearizations for which the given coloring of P is still a bicoloring. Let us call such a linearization *compatible* with the given bicoloring. Then the above discussion yields

Theorem 4. *If* $P = (|P|, \preccurlyeq)$ *is a finite bicolored poset, then the sign-imbalance of the set of all linearizations of* P *(relative to any fixed indexing of* $|P|$*) is equal to the sign-imbalance (relative to the same indexing) of the set of linearizations compatible with the given bicoloring.*

In particular, if P *has no linearizations compatible with the given bicoloring, then it is sign-balanced.* □

Easy examples of bicolored posets P with no compatible linearizations are those in which the difference between the numbers of elements of the two colors is greater than 1. If a poset P has only one connected component, then a bicoloring of P, if one exists, is unique up to interchange of colors; an example in which this essentially unique bicoloring has the property just mentioned is , with three elements of one color and just one of the other. So that poset is sign-balanced.

If a bicolorable poset has more than one connected component, we have a larger choice of bicolorings. For instance, if P consists of two connected components, each of which is a chain with an odd number of elements, and we bicolor those chains so that their least elements have the same color, then we find that the difference between the total numbers of elements of the two colors is 2; so this poset, too, is sign-balanced. If instead we color the two chains so that their least elements have opposite colors, then P has the same number of elements of each color; so though we have just seen that it is sign-balanced, this second bicoloring does not give a proof of that fact.

Examples of *non*-sign-balanced posets are and ; as one can quickly check by counting.

Theorem 4 allows us to give an alternative proof of the following result of R. Stanley. (The formulation of that result in [5] refers to the *length* of chains, i.e., the number of *steps*, which is one less than the cardinality of the chain, in terms of which I state it here.)

Lemma 5 (R. Stanley [5, Corollary 2.2])**.** *If a finite poset* $P = (|P|, \preccurlyeq)$ *has the property that the cardinalities of its maximal chains all have the same parity, and this is the opposite of the parity of the cardinality of* P*, then* P *is sign-balanced.*

Proof. The fact that the cardinalities of all maximal chains of P have the same parity implies in particular that for each $x \in |P|$, the cardinalities of all maximal chains in the downset generated by x have a common parity. (Otherwise, by combining two such chains of different parities with a common maximal chain in the upset generated by x, one would get two maximal chains in P of different parities.) Classifying elements x according to whether maximal chains below x all have odd or even cardinality, we get a bicoloring of P, under which all minimal elements have a common color. From this we see that if the common parity of the cardinalities of all maximal chains of P is odd, the maximal elements of P will all be of the same color as the minimal elements, while if it is even, they will all have the other color.

Suppose now that P has a linearization compatible with the above bicoloring. Because the parity of P is the opposite of that of its maximal chains, the relationship between the colors of the elements at the bottom and top of this linearization (same-color versus different-color) will be the opposite of what is the case for maximal chains of P.

On the other hand, the bottom element of the linearization must be the bottom element of one of those maximal chains, and hence must be of the color which all those bottom elements have, and the top element must similarly be the color of the top elements of all the maximal chains, so, in contradiction to the preceding paragraph, the relationship between the colors of elements at the top and bottom of this linearization must be the same as for maximal chains of P. This contradiction shows that no compatible linearization of P can exist; so by Theorem 4, P is sign-balanced. □

An easy example to which the above lemma applies is that in which P has two connected components, each a chain of odd cardinality (for which we got the same conclusion in the second paragraph after Theorem 4). Two more easy examples are the posets and .

3. Bringing sign-imbalance and linearization-count together

If P is a finite poset, let $L_0(P)$ denote the number of even linearizations of P, and $L_1(P)$ the number of odd linearizations (relative to some fixed linearization). Then it is natural to associate to P the element $L(P) = L_0(P) + L_1(P)\,\zeta$ of the group ring $\mathbb{Z}\{1,\zeta\}$, where $\{1,\zeta\}$ is the 2-element group, written multiplicatively.

$\mathbb{Z}\{1,\zeta\}$ admits two ring homomorphisms to $\mathbb{Z}$, taking ζ to $+1$ and -1 respectively; thus, the former carries $L(P)$ to $L_0(P) + L_1(P)$, the number of linearizations of P, which we shall denote $L_+(P)$, while the latter carries it to $L_0(P) - L_1(P)$, the sign-imbalance of P, which we shall denote $L_-(P)$. These homomorphisms together yield an embedding of $\mathbb{Z}\{1,\zeta\}$ in $\mathbb{Z}\times\mathbb{Z}$, whose image is $\{(a,b) \in \mathbb{Z}\times\mathbb{Z} \mid a \equiv b \pmod 2\}$. The image of $L(P) = L_0(P) + L_1(P)\,\zeta$ under this embedding, $(L_+(P), L_-(P))$, will be denoted $L_\pm(P)$.

For any finite poset P, the element $L(P)$ lies in the subsemiring $\mathbb{N}\{1,\zeta\}$ of $\mathbb{Z}\{1,\zeta\}$ consisting of elements in which the coefficients of 1 and ζ are both nonnegative. The image of this semiring under the above ring embedding is

$$\{(a,b) \in \mathbb{Z}\times\mathbb{Z} \mid a \equiv b \pmod 2 \text{ and } |b| \le a\}. \tag{3}$$

It will follow from Theorem 7 below that for various finite posets P and P', we can say that the number of linearizations $L_+(P)$ is a multiple of $L_+(P')$. (For instance, this is true if P' is any of the connected components of P.) In such situations, it is not surprising to find that the sign-imbalance $L_-(P)$ is, likewise, a multiple of $L_-(P')$. But in fact, one typically has the stronger statement that $L(P)$ is a multiple of $L(P')$ in $\mathbb{N}\{1,\zeta\}$, equivalently, that the ordered pair $L_\pm(P)$ is a multiple of $L_\pm(P')$ in the semiring (3).

So, for instance, if $L_\pm(P') = (4,2)$, then $L_\pm(P)$ cannot be $(12,0)$ or $(8,2)$ or $(8,8)$. Separate consideration of the numbers $L_+(P)$ and $L_-(P)$ would not exclude these values; but the pairs of integers by which $(4,2)$ would have to be multiplied to get them would be $(3,0)$, $(2,1)$ and $(2,4)$, of which the first two don't have coordinates of the same parity, while the last fails to satisfy the inequality $|b| \le a$ of (3). (The element $(2,4) \in \mathbb{Z}\times\mathbb{Z}$ corresponds to $3-\zeta \notin \mathbb{N}\{1,\zeta\}$.)

We will, in the next three sections, consider L_+ and L_- together. We will encounter both parallelisms and contrasts between their behaviors.

We remark that the element $L(P) \in \mathbb{N}\{1,\zeta\}$ defined above is the image, under the homomorphism taking q to ζ, of the polynomial $I_{P,\omega}(q)$ defined by Stanley in [5, p.881, display (1)]. That polynomial

depends in a much stronger way than $L(P)$ on the reference linearization (expressed there by an indexing ω of $|P|$), and, as Stanley notes, does not seem easy to understand.

4. The linearization-count and sign-imbalance of a lexicographic sum

4.1. **Review of lexicographic sums.** Suppose $P_0 = (|P_0|, \preccurlyeq_0)$ is a poset, and that for each $x \in |P_0|$ we are given a poset $P_x = (|P_x|, \preccurlyeq_x)$. Then we can form the disjoint union of the sets $|P_x|$, say constructed as $\{(x, x') \mid x \in |P_0|,\ x' \in |P_x|\}$, and give this set a partial ordering under which, for each $x \in |P_0|$, the copy $\{(x, x') \mid x' \in |P_x|\}$ of $|P_x|$ is made order-isomorphic to P_x via the correspondence $(x, x') \mapsto x'$, while for $x \neq y$ in $|P_0|$, the order-relation between elements (x, x') and (y, y') $(x' \in |P_x|,\ y' \in |P_y|)$ is determined solely by their first components: $(x, x') \preccurlyeq (y, y')$ if and only if $x \preccurlyeq_0 y$.

For finite posets, considered below, we shall find it convenient to assume $|P_0|$ indexed as $\{x_1, \ldots, x_{m_0}\}$. We shall then abbreviate $P_{x_i} = (|P_{x_i}|, \preccurlyeq_{x_i})$ to $P_i = (|P_i|, \preccurlyeq_i)$, and assume that each $|P_i|$ is indexed as $\{x_{i,1}, \ldots, x_{i,m_i}\}$, with distinct symbols $x_{i,j}$ $(i \leq m_0,\ j \leq m_i)$ denoting distinct elements. This makes the $|P_i|$ disjoint, so we can forego the construction of their disjoint union by ordered pairs, and simply take the underlying set of our lexicographic sum to be their union. Choosing a notation for such a lexicographic sum, and summarizing the above description of it, we have

$$\begin{aligned} & P_0 * (P_1, \ldots, P_{m_0}) \ = \ (|P|, \preccurlyeq) \ = \ P, \text{ where} \\ & |P| \ = \ \{x_{i,j} \mid 1 \leq i \leq m_0,\ 1 \leq j \leq m_i\}, \text{ and} \\ & x_{i,j} \preccurlyeq x_{i',j'} \text{ if and only if either} \\ & \quad i \neq i' \text{ and } x_i \preccurlyeq_0 x_{i'} \text{ (in } P_0\text{), or} \\ & \quad i = i' \text{ and } x_{i,j} \preccurlyeq_i x_{i,j'} \text{ (in } P_i\text{).} \end{aligned} \tag{4}$$

Two easy classes of examples: A poset P decomposed into its connected components P_i can be regarded as the lexicographic sum of the P_i over an antichain P_0; in particular, the next-to-last steps in the construction of section 1 were lexicographic sums of chains over antichains. On the other hand, the "prelinearizations" that occurred at the next-to-last step in section 2 were lexicographic sums of antichains over chains.

In a lexicographic sum $P_0 * (P_1, \ldots, P_{m_0})$, some or all of the P_i may be empty. Occasionally, such cases will require special consideration.

Let us make

Convention 6. *In the context of* (4)*, for the purpose of defining the parities of linearizations, we define on each* $|P_i|$ $(0 \leq i \leq m_0)$ *the linear* reference order $x_{i,1} \preccurlyeq_{i,\mathrm{ch}} \ldots \preccurlyeq_{i,\mathrm{ch}} x_{i,m_i}$ (*where* ch *is mnemonic for "chain"*)*, and for each* i *we shall write* $P_{i,\mathrm{ch}} = (|P_i|, \preccurlyeq_{\mathrm{ch}})$.

On $|P_0 * (P_1, \ldots, P_{m_0})|$ *we likewise define the reference order*

$$x_{1,1} \preccurlyeq_{\mathrm{ch}} \ldots \preccurlyeq_{\mathrm{ch}} x_{1,m_1} \preccurlyeq_{\mathrm{ch}} \ \ldots \ \preccurlyeq_{\mathrm{ch}} x_{m_0,1} \preccurlyeq_{\mathrm{ch}} \ldots \preccurlyeq_{\mathrm{ch}} x_{m_0,m_{m_0}}. \tag{5}$$

4.2. **Computing L of a lexicographic sum.** Sections 1 and 2 both used the idea, "linearize the pieces, and see how you can combine the resulting chains". This idea is abstracted in the following easy result.

Theorem 7. *In the context of* (4) *and Convention 6, we have*

$$L(P_0 * (P_1, \ldots, P_{m_0})) \ = \ \Big(\textstyle\prod_{1 \leq i \leq m_0} L(P_i)\Big) \cdot L(P_0 * (P_{1,\mathrm{ch}}, \ldots, P_{m_0,\mathrm{ch}})). \tag{6}$$

(*Hence, the same is true with* L *everywhere replaced by* $L_\pm$*, or by* L_+*, or by* L_-.)

Sketch of proof. To obtain the general linearization of the poset $P_0 * (P_1, \ldots, P_{m_0}) = (|P|, \preccurlyeq)$, let us first choose the linear order to be used on each $|P_i|$, which can be any linearization of $\preccurlyeq_i$, then specify how the union of the resulting chains is to be ordered. By the definition of lexicographic sum, the order-relation in $P_0 * (P_1, \ldots, P_{m_0}) = (|P|, \preccurlyeq)$ between any element of P_i and any element of P_j with $i \neq j$, depends only on i and j; hence for each way of linearizing the separate P_i, the ways of linearizing their union which are compatible with those linearizations and with the ordering of $P_0 * (P_1, \ldots, P_{m_0}) = (|P|, \preccurlyeq)$ correspond to the ways of linearizing a lexicographic sum over P_0 of a family of chains of the corresponding lengths, of which $P_0 * (P_{1,\mathrm{ch}}, \ldots, P_{m_0,\mathrm{ch}})$ is one. It is also not hard to check that the parity of a linearization of $P_0 * (P_1, \ldots, P_{m_0})$ (relative to the reference-ordering $\preccurlyeq_{\mathrm{ch}}$) is the product of the parities of the induced linearizations of the P_i (relative to the reference-orderings $\preccurlyeq_{i,\mathrm{ch}}$) and that of the corresponding linearization of $P_0 * (P_{1,\mathrm{ch}}, \ldots, P_{m_0,\mathrm{ch}})$ (again relative to $\preccurlyeq_{\mathrm{ch}}$). The equality (6) follows.

The final parenthetical assertion follows because the operator $L_\pm$ is the composite of the operator L with a ring homomorphism $\mathbb{Z}\{1,\zeta\} \to \mathbb{Z}\times\mathbb{Z}$, and L_+, L_- are in turn the composites of $L_\pm$ with the projection homomorphisms $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$. □

4.3. **Results on lexicographic sums of chains.** For Theorem 7 to be useful, we need to be able to compute $L(P)$ for P a lexicographic sum of *chains.* We obtain results in this direction below. Since in these results, the chains in question are not given as listings of the elements of other posets, we adjust our notation slightly.

Definition 8. *Let $P_0 = (|P_0|, \preccurlyeq_0)$ be a finite poset, with $|P_0| = \{x_1,\dots,x_{m_0}\}$. Then for nonnegative integers $m_1,\dots,m_{m_0}$, we define*

$$L(P_0;\, m_1,\dots,m_{m_0}) \;=\; L(P_0 * (C_1,\dots,C_{m_0})) \tag{7}$$

where for $1\le i\le m_0$, $C_i = (|C_i|, \preccurlyeq_i)$ is a chain of m_i elements $x_{i,1}\preccurlyeq_i \dots \preccurlyeq_i x_{i,m_i}$, these chains being understood to be pairwise disjoint, and the parities of linearizations of their lexicographic sum being taken relative to the reference ordering (5)

We will also use the notation corresponding to (7) *with L_0, L_1, L_+, L_- and $L_\pm$ in place of L.*

A trivial case is

Proposition 9. *Suppose, in Definition 8, that P_0 is a chain, $x_1\preccurlyeq_0 \dots \preccurlyeq_0 x_{m_0}$. Then for all $m_1,\dots,m_{m_0}$, $L(P_0;\, m_1,\dots,m_{m_0}) = 1$. Equivalently, $L_+(P_0;\, m_1,\dots,m_{m_0}) = L_-(P_0;\, m_1,\dots,m_{m_0}) = 1$.*

Proof. With P_0 and all of $C_1,\dots,C_{m_0}$ being chains, we see that $P_0 * (C_1,\dots,C_{m_0})$ is a chain, so it has just one linearization. This is our reference ordering, so it has even parity, giving our first conclusion, which is equivalent to the final equation. □

If P_0 is an antichain we can also get an exact result. In the proofs of Proposition 10 below, and of Theorem 11, which will generalize the hard part of that result, we shall, following [5], use the imagery of "dominoes". Namely, to partly or completely cover a poset P with dominoes means to distinguish certain *non-overlapping* pairs of elements (x,y) such that y covers x (i.e., x and y are distinct elements, such that $x\preccurlyeq y$ and there are no elements strictly between these). If (x,y) is a pair which we have so distinguished, we will speak of there being a domino lying over x and y.

(Richard Stanley has pointed out to me that Proposition 10 is essentially known: Parts (i) and (ii) can be obtained by setting $q=-1$ in formula (1.68) of [6], and calling on (1.66) and (1.87) thereof.)

Proposition 10. *Suppose, in the context of Definition 8, that P_0 is an antichain.*

Then the linearization count $L_+(P_0;\, m_1,\dots,m_{m_0})$ is the multinomial coefficient

$$\textstyle\big(\sum_{1\le i\le m_0} m_i\big)!\,/\,\prod_{1\le i\le m_0} m_i!\,. \tag{8}$$

On the other hand, the sign-imbalance $L_-(P_0;\, m_1,\dots,m_{m_0})$ is described as follows:

(i) *If at most one of $m_1,\dots,m_{m_0}$ is odd, then $L_-(P_0;\, m_1,\dots,m_{m_0})$ is the multinomial coefficient*

$$\textstyle\big(\sum_{1\le i\le m_0} \lfloor m_i/2\rfloor\big)!\,/\,\prod_{1\le i\le m_0} \lfloor m_i/2\rfloor!\,. \tag{9}$$

(ii) *If more than one of $m_1,\dots,m_{m_0}$ are odd, then $L_-(P_0;\, m_1,\dots,m_{m_0}) = 0$.*

Proof. The number $L_+(P_0;\, m_1,\dots,m_{m_0})$ of linearizations of our lexicographic sum of chains is the number of ways of partitioning the $\sum_{1\le i\le m_0} m_i$ positions available in such a linearization into subsets of cardinalities $m_1,\dots,m_{m_0}$, and (8) is a standard description of this number [6, p.20].

To get information on $L_-(P_0;\, m_1,\dots,m_{m_0})$, let us cover as much as we can of each linearization of $P_0 * (C_1,\dots,C_{m_0})$ with dominoes, starting from the bottom. Thus, if $\sum m_i$ is even, each linearization of our lexicographic sum will be entirely covered, while if it is odd, all but the top element of each linearization will be covered.

I claim that the set of linearizations which, when dominoes are so placed, have the property that at least one domino lies over two elements that are *incomparable* under $\preccurlyeq$ is sign-balanced. Indeed, let us pair off these linearizations as follows. Given such a linearization, find, among dominoes that lie over $\preccurlyeq$-incomparable pairs, the top one, and form a new linearization by reversing the positions of the pair of elements it lies over. Because those elements are $\preccurlyeq$-incomparable, the result will again be a linearization of $\preccurlyeq$; and because exactly one pair has been interchanged, the new linear order has parity opposite to the old one. It is also

clear that the above construction, applied to the new linearization, returns the original one, so we indeed have a pairing. Hence the contributions to the sign-imbalance of those linearizations sum to zero.

Thus, to determine the sign-imbalance of $P_0 * (C_1, \ldots, C_{m_0})$, it suffices to consider linearizations which, when dominoes are set down as above, have the property that every domino lies over a pair of *comparable* elements. Let us call such a linearization of $\preccurlyeq$ *distinguished*.

Since elements from different chains C_i are incomparable, a distinguished linearization will have the property that every domino lies over a pair of elements from the same C_i. More precisely, we see by induction, working up from the bottom, that every domino will lie over a pair of the form $(x_{i,2j-1},\, x_{i,2j})$. From this it easily follows that any distinguished linearization can be turned into our reference ordering (5) by repeatedly switching the places of two adjacent dominoes (hence, reordering a string $w < x < y < z$ as $y < z < w < x$), or moving a domino past the lone dominoless element, if there is one (reordering a string $x < y < z$ as $z < x < y$.) A movement of either sort can be done using 2 transpositions of elements, so in each case, the parity of the linearization does not change. Hence every distinguished linearization is even, so the sign-imbalance of $P_0 * (C_1, \ldots, C_{m_0})$ equals the number of distinguished linearizations.

From the above considerations, we can see that if there exists a distinguished linearization, at most one of the C_i can have odd cardinality (the C_i to which the "lone element", if any, at the top of a distinguished linearization belongs). This immediately gives statement (ii).

To get statement (i), suppose first that all the m_i are even. Then it is not hard to see that each distinguished linearization is determined by noting how the $(\sum m_i)/2$ dominoes are partitioned into $m_1/2$ dominoes lying over pairs of members of C_1, $m_2/2$ lying over pairs of members of C_2, etc.. By the same counting principle used in getting (8), the number of such partitions is $(\sum m_i/2)\,!/\prod (m_i/2)\,!$, which agrees with (9) in this case.

Now suppose instead that exactly one of the m_i, say m_{i_0}, is odd. Then in a distinguished linearization, the one element not under any domino, namely, the top element of our linearization, is necessarily the largest element of the chain C_{i_0}. So a distinguished linearization will be determined by the arrangement of the dominoes covering the remaining elements, and as before, the number of possibilities is described by (9). □

For general P_0, we do not have an exact formula for $L_+(P_0;\, m_1, \ldots, m_{m_0})$; but we can again get a result on $L_-(P_0;\, m_1, \ldots, m_{m_0})$, in the spirit of the above proposition.

Theorem 11. *Given P_0 as in Definition 8, and nonnegative integers $m_1, \ldots, m_{m_0}$, let*

$$S \;=\; \{x_i \mid m_i \text{ is odd}\} \;\subseteq\; |P_0|. \tag{10}$$

Then

(i) *If S forms a chain under $\preccurlyeq_0$ (possibly the empty chain), then we have*

$$L_-(P_0;\, m_1, \ldots, m_{m_0}) \;=\; \pm\, L_+(P_0;\, \lfloor m_1/2 \rfloor, \ldots, \lfloor m_{m_0}/2 \rfloor), \tag{11}$$

with the $+$ sign applying if the reference ordering of P_0 is isotone on S.

(ii) *If, on the other hand, S contains two elements x_{i_0} and x_{i_1} which are incomparable, and which* are majorized by *exactly the same set of other elements of P_0, or alternatively, which* majorize *exactly the same set of other elements of P_0, then*

$$L_-(P_0;\, m_1, \ldots, m_{m_0}) \;=\; 0. \tag{12}$$

Proof. Let us place dominoes on each linearization of $P_0 * (C_1, \ldots, C_{m_0})$, say with ordering $\leq$, as follows, working upward from the bottom.

Suppose inductively that we have specified where dominoes are to be placed on our linearization up to a certain point, but not all the way to the top. Let $x_{i,j}$ be the least element such that we have not specified whether a domino is to be placed on it. If $x_{i,j}$ is the greatest element of our linearization, then clearly we can put down no more dominoes; in particular, $x_{i,j}$ will remain uncovered. If, rather, $x_{i,j}$ is followed immediately either by another element from same chain C_i, or by an element which belongs to a different chain $C_{i'}$, and is $\preccurlyeq$-incomparable with $x_{i,j}$, let us put a domino over this pair of elements. Finally, if $x_{i,j}$ is followed in our linearization by an element $x_{i',j'}$ with $i' \neq i$ and $x_i \preccurlyeq_0 x_{i'}$, we put no domino over $x_{i,j}$ (so that our recursive construction will continue with $x_{i',j'}$ in the role that $x_{i,j}$ had).

Having placed dominoes in this manner on every linearization of $P_0 * (C_1, \ldots, C_{m_0})$, we note that, as in the proof of Proposition 10, if a linearization has a domino placed over two incomparable elements, then

looking at the highest domino with this property, and reversing the order of the elements under it, we get another linearization, of the opposite parity, on which the dominoes will have been placed over the same pairs of elements. This again pairs off linearizations of opposite parities, hence we can again ignore linearizations so paired in determining $L_-(P_0;\, m_1, \ldots, m_{m_0})$.

Let us again call those linearizations which are not so paired, i.e., which involve dominoes only over pairs $(x_{i,j},\, x_{i,j+1})$ of elements of the same chain C_i, "distinguished".

Note that the way we have placed dominoes on our linearizations insures that if an element $x_{i,j}$ is not under a domino, then it must be the largest element of the chain C_i to which it belongs, namely, x_{i,m_i}. (Indeed, otherwise, the larger elements of C_i would lie above it, so it could not be the top element of the linearization; and they would lie between it and the elements of any $C_{i'}$ with $x_i \preccurlyeq_0 x_{i'}$, so it would not get skipped in the process of setting down dominoes.) Clearly, also, if such an element is not the largest element in the whole linearization, then the element that follows it must have the form $x_{i',1}$. Finally, if our linearization is distinguished, then all pairs of elements under dominoes must have the form $(x_{i,j},\, x_{i,j+1})$; and for each i, we again see by induction that dominoes will cover precisely the pairs $(x_{i,2j-1}, x_{i,2j})$ for $1 \leq j \leq \lfloor m_i/2 \rfloor$. Thus, the top element x_{i,m_i} of a chain C_i will not lie under a domino if and only if m_i is odd.

Now in every distinguished linearization of $P_0 * (C_1, \ldots, C_{m_0})$, the $\sum_{1 \leq i \leq m_0} \lfloor m_i/2 \rfloor$ dominoes must be arranged in one of the $L_+(P_0; \lfloor m_1/2 \rfloor, \ldots, \lfloor m_{m_0}/2 \rfloor)$ ways corresponding to linearizations of the lexicographic sum over P_0 of a family of chains of lengths $\lfloor m_i/2 \rfloor$. I claim that if, as in the hypothesis of statement (i), the set S defined in (10) forms a chain in P_0, then (a) each of the above arrangements of dominoes appears in one and only one distinguished linearization of $P_0 * (C_1, \ldots, C_{m_0})$, and (b) if the reference ordering of P_0 is isotone on S, then all distinguished linearizations have even parity. This will prove (i).

To see assertion (a), assume we are given a linearization of the indicated family of dominoes, and let us see why there is a unique way of placing the "lone" elements x_{i,m_i} $(x_i \in S)$ among them so as to get a distinguished linearization of P. Note that since S is a chain in P_0, the *relative* ordering of those lone elements is predetermined.

We first describe the unique position at which the *highest* lone element x_{i,m_i} can appear in such a linearization: If there is no $x_{i'}$ in P_0 with $x_i \preccurlyeq_0 x_{i'}$, then by our description of distinguished linearizations, x_{i,m_i} can only appear at the very top; while if there are such elements $x_{i'}$, they will necessarily belong to $|P_0| - S$, and x_{i,m_i} will necessarily appear just below the element $x_{i',1}$ with $x_i \preccurlyeq_0 x_{i'}$ that appears lowest among such elements in our given domino-covered chain.

We now repeat this idea, working down the chain of elements $x_i \in S$: for each x_i, we place x_{i,m_i} just below the lowest element $x_{i',1}$ such that $x_i \preccurlyeq_0 x_{i'}$ that we have already positioned. (This $x_{i',1}$ may or may not lie under a domino. It will not if $x_{i'} \in S$ and $m_{i'}$ happens to be 1; but in this case, by the order in which we have chosen to do things, it will be an element whose position we have already determined.) It is easy to see that the resulting ordering is indeed a linearization of $P_0 * (C_1, \ldots, C_{m_0})$, in fact a distinguished linearization, and is the only distinguished linearization compatible with the given arrangement of dominoes.

Assuming now that the reference ordering of P_0 is isotone on S, we see that each of the distinguished linearizations described above can be obtained from the reference linearization of $P_0 * (C_1, \ldots, C_{m_0})$ by a series of steps, each of which either moves one domino past another, or moves a domino past a lone element. (By our assumption that the reference ordering of P_0 is isotone on S, we never have to move one lone element past another.) Again, each of these steps acts by an even permutation, so our distinguished orderings indeed have even parity, proving (b), and hence proving (i) in the case where the reference ordering is isotone on S. In the contrary case, a change in the reference ordering will multiply $L_-(P_0;\, m_1, \ldots, m_{m_0})$ by ± 1, giving the general case of (i).

Turning to (ii), suppose first that S contains incomparable elements x_{i_0} and x_{i_1} which are majorized by the same sets of other elements of P_0. Then in a distinguished linearization of $P_0 * (C_1, \ldots, C_{m_0})$, the descriptions of where the elements $x_{i_0,m_{i_0}}$ and $x_{i_1,m_{i_1}}$ must occur are the same – at the top if the set of elements $x_i \in |P_0|$ strictly greater than x_{i_0} (equivalently, strictly greater than x_{i_1}) is empty; otherwise, immediately below the lowest element $x_{i,1}$ of the linearization such that x_i which is strictly greater than these elements. But $x_{i_0,m_{i_0}}$ and $x_{i_1,m_{i_1}}$ cannot both be in that position, so $P_0 * (C_1, \ldots, C_{m_0})$ has *no* distinguished linearizations, and (12) follows.

If, rather, S has incomparable elements x_{i_0} and x_{i_1} which *majorize* the same set of other elements of P_0, the result follows from the above case by the invariance of the property of sign-balance under reversal of order. This completes the proof of (ii).

(Since our rule for placing dominoes, and hence our concept of distinguished linearization, are *not* invariant under reversal of order, we can't claim in the last case that $P_0 * (C_1, \ldots, C_{m_0})$ has no distinguished linearizations. Rather, a direct proof of (12) for that case would involve defining "reverse-distinguished" linearizations, and showing that $P_0 * (C_1, \ldots, C_{m_0})$ has none of these.) $\square$

If we apply the above theorem to the case where P_0 is an antichain, then every subset $S \subseteq |P_0|$ clearly falls under one of cases (i) or (ii) above, depending on whether it has ≤ 1 or > 1 elements, and we get the description of $L_-(P_0;\, m_1, \ldots, m_{m_0})$ in Proposition 10. The simplest examples of posets P_0 for which not every subset S is covered by our theorem are the 4-element posets and , with S consisting, in each case, of the lower left and upper right elements. In the former case, the function $L_\pm(P_0;\, m_1, m_2, m_3, m_4)$ is nevertheless easily evaluated. The second case looks harder, but I have not examined it closely.

5. Some properties of $L_\pm(P_0;\, m_1, \ldots, m_{m_0})$ as a function of $m_0, \ldots, m_{m_0}$

5.1. **Motivation: the case where P_0 is a 2-element antichain.** Let us consider the simplest nontrivial case of Proposition 10, where $m_0 = 2$. Then $P_0 * (C_1, C_2)$ is the disconnected union of a chain of m_1 elements and a chain of m_2 elements, so its linearizations correspond to the ways of partitioning a chain of $m_1 + m_2$ elements into two sets, of m_1 and m_2 elements respectively. The number of these is the binomial coefficient $\binom{m_1+m_2}{m_2}$, so the values of $L_+(P_0;\, m_1, m_2)$ are the entries of Pascal's triangle. We display in (13) the first few rows of that triangle, and likewise, the values of $L_-(P_0;\, m_1, m_2)$ given by that same proposition. In each array in (13), the rows correspond to the values of $m_1 + m_2$, the diagonals going downward to the left to the values of m_1, and the diagonals going downward to the right to the values of m_2.

(13)

$L_+(P_0;\, m_1, m_2)$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$L_-(P_0;\, m_1, m_2)$

1								
1	1							
1	0	1						
1	1	1	1					
1	0	2	0	1				
1	1	2	2	1	1			
1	0	3	0	3	0	1		
1	1	3	3	3	3	1	1	
1	0	4	0	6	0	4	0	1

The familiar rule for producing Pascal's triangle, that each entry is the sum of the two above it in the preceding row, can be interpreted in terms of $L_+(P_0;\, m_1, m_2)$: If we classify linearizations of $P_0 * (C_1, C_2)$ according to whether the top element belongs to C_1 or C_2, then in the former case, the ordering of the remaining elements constitutes a linearization of the union of a chain of $m_1 - 1$ elements and a chain of m_2 elements, in the latter case, a linearization of a union of a chain of m_1 elements and a chain of $m_2 - 1$ elements.

The sign-imbalances $L_-(P_0;\, m_1, m_2)$ turn out to satisfy a similar law: every entry that lies an *even* number of steps from the left-sloping edge of the array is, as in Pascal's triangle, the sum of the two entries above it, while if an entry lies an odd number of steps from that edge, it is the *difference* of those two entries (the one to the left minus the one to the right). The reader can justify this rule by classifying linearizations of our union of chains as in the preceding paragraph, and examining how the parity of a linearization compares with the parity of the linearization of the one-element-smaller poset that we get on dropping the element at the top of that linearization.

(In fact, I discovered the formulas of Proposition 10(i)-(ii) by first studying the case $m_0 = 2$, and obtaining the above recursive rule for $L_-(P_0;\, m_1, m_2)$, then noticing the way values from Pascal's triangle occurred in the resulting array, and thinking about how to justify that pattern.)

5.2. $L_\pm(P_0; m_1, \dots, m_{m_0})$ **and one-variable polynomials.** In general, $L_\pm(P_0; m_1, \dots, m_{m_0})$ is not a polynomial function of $m_1, \dots, m_{m_0}$. For instance, if P_0 is a 2-element antichain as in the preceding subsection, we have $L_+(P_0; m, m) = \binom{2m}{m} \geq 2^m$, so it grows too fast to be polynomial. But note also that if, in $L_+(P_0; m_1, m_2) = \binom{m_1+m_2}{m_1}$, we hold m_2 constant, the resulting function is a polynomial in m_1 (of degree m_2), while if we hold m_1 constant, it is a polynomial in m_2 (of degree m_1). Here is a similar result about L_+ for general P_0, and a somewhat more complicated statement about L_-.

Proposition 12. *Let $P_0 = (|P_0|, \preccurlyeq_0)$ be a finite poset, with $|P_0| = \{x_1, \dots, x_{m_0}\}$, let $i_0 \in \{1, \dots, m_0\}$, and for each $i \neq i_0$ in $\{1, \dots, m_0\}$, let us fix a value for m_i. Then, regarding $L_\pm(P_0; m_1, \dots, m_{m_0})$ as a function of m_{i_0}, we have*

(a) *$L_+(P_0; m_1, \dots, m_{m_0})$ is a polynomial in m_{i_0}, whose degree is the sum of those values of m_i $(i \neq i_0)$ such that x_i is $\preccurlyeq_0$-incomparable with x_{i_0}.*

(b) *The values of $L_-(P_0; m_1, \dots, m_{m_0})$ are given by two polynomials in m_{i_0}, one for m_{i_0} even, the other for m_{i_0} odd. Each has degree less than or equal to $\lfloor \sum m_i/2 \rfloor$, where the summation is again over those i such that x_i is $\preccurlyeq_0$-incomparable with x_{i_0}.*

Proof. Again let $C_1, \dots, C_{m_0}$ be disjoint chains of lengths $m_1, \dots, m_{m_0}$. Then a linearization of $P = P_0 * (C_1, \dots, C_{m_0})$ can be determined by first choosing an arbitrary linearization of the subposet with underlying set $|P| - |C_{i_0}|$, then specifying where to insert the elements of $|C_{i_0}|$.

To see where those elements can go, let us partition $\{1, \dots, m_0\} - \{i_0\}$ into three subsets: the set $I_<$ of those i such that $x_i \preccurlyeq_0 x_{i_0}$, the set $I_>$ of those i such that $x_{i_0} \preccurlyeq_0 x_i$, and the set $I_\sim$ of those i such that x_i is $\preccurlyeq_0$-incomparable with x_{i_0}. Then given any linearization $\leq$ of $\preccurlyeq$ on $|P| - |C_{i_0}|$, the elements of $|C_{i_0}|$ can be inserted within the range bounded below by the highest location, under that linearization, of an element of $\bigcup_{i \in I_<} |C_i|$, and above by the lowest location of an element of $\bigcup_{i \in I_>} |C_i|$; where we understand the former restriction to be vacuous if $\bigcup_{i \in I_<} |C_i|$ is empty, and the latter if $\bigcup_{i \in I_>} |C_i|$ is empty.

In our linearization of $|P| - |C_{i_0}|$, the interval we have described will be populated by some subset (possibly empty) of $\bigcup_{i \in I_\sim} |C_i|$. If d is the number of elements of that set in that interval, then clearly

$$0 \leq d \leq \textstyle\sum_{i \in I_\sim} m_i, \tag{14}$$

and we see that the number of ways the m_{i_0} elements of the chain C_{i_0} can be interspersed among those d elements is $\binom{m_{i_0}+d}{d}$, which, as a function of m_{i_0}, is a polynomial of degree d. The maximum value allowed by (14), $d = \sum_{i \in I_\sim} m_i$, does in fact occur, since we can linearize $|P| - |C_{i_0}|$ so that all members of $\bigcup_{i \in I_<} |C_i|$ precede all members of $\bigcup_{i \in I_\sim} |C_i|$, and these precede all members of $\bigcup_{i \in I_>} |C_i|$. Summing the polynomials in m_{i_0} obtained from all our linearizations of $|P| - |C_{i_0}|$, we get a polynomial $f(m_{i_0})$ describing $L_+(P_0; m_1, \dots, m_d)$. Since the leading coefficients of the polynomials we have summed are all positive, the terms of degree $\sum_{i \in I_\sim} m_i$ cannot cancel, so that is the degree of $f(m_{i_0})$, completing the proof of (a).

If, instead, we look at the sign-imbalance, then the polynomials $\binom{m_{i_0}+d}{d}$ are replaced by the functions described in parts (i) and (ii) of Proposition 10 for $m_0 = 2$ (the diagonals of the right-hand array of (13)), multiplied by ± 1 depending on the details of our linearization of $|P| - |C_{i_0}|$. Each of these is given by one polynomial of degree $\leq d/2$ (possibly the zero polynomial) on odd inputs and another polynomial of degree $\leq d/2$ on even inputs, yielding (b). Because of the varying signs, we cannot say in this case that the leading terms of the highest-degree polynomials will not cancel, hence we cannot specify the exact degrees of the polynomials we get. □

Comparing with Proposition 10, we might wonder whether, of the two polynomials referred to in (b) above, the one that gives the sign-imbalance for m_{i_0} odd must have degree less than or equal to that of the one that does so for m_{i_0} even. But this is not the case. For instance, suppose P_0 is the poset $x_1 \bullet \begin{smallmatrix} x_3 \\ x_2 \end{smallmatrix}$, let $i_0 = 2$, and take m_1 and m_3 odd. Then for any value of m_2, the union of the posets C_2 and C_3 will be a chain of length $m_2 + m_3$, which has the opposite of the parity of m_2, whence we see from Proposition 10 that for m_2 even, $L_-(P_0; m_1, m_2, m_3)$ will be zero, while for m_2 odd, it will be a polynomial of degree $\lfloor m_1/2 \rfloor$.

5.3. **Chains in P_0 again.** At the beginning of the preceding subsection, we noted that for P_0 a 2-element antichain, the function $L_+(P_0; m_1, m_2)$ was not a polynomial in its two variables. More generally, if x_{i_1} and

x_{i_2} are incomparable elements of a finite poset P_0, and we fix values for all m_i other than m_{i_1} and m_{i_2}, we find that as a function of those two variables, $L_+(P_0;\, m_1, \ldots, m_{m_0})$ cannot be a polynomial. Indeed, one can strengthen the ordering of P_0 to get an "almost-chain" P_0', in which the only pair of incomparable elements is $\{x_{i_1},\, x_{i_2}\}$. Then $L_+(P_0;\, m_1, \ldots, m_{m_0}) \geq L_+(P_0';\, m_1, \ldots, m_{m_0}) = \binom{m_{i_1}+m_{i_2}}{m_{i_1}}$. But again, for $m_{i_1} = m_{i_2} = m$ we have $\binom{m_{i_1}+m_{i_2}}{m_{i_1}} = \binom{2m}{m} \geq 2^m$, which grows too rapidly for a polynomial function. It follows that $L_+(P_0;\, m_1, \ldots, m_{m_0})$ is not a polynomial function in any subset of its arguments two of which correspond to incomparable elements of P_0.

Can $L_+(P_0;\, m_1, \ldots, m_{m_0})$ ever be a polynomial function of more than one of its arguments? Yes – in precisely the cases not excluded by the above observations:

Theorem 13. *Let $P_0 = (|P_0|, \preccurlyeq_0)$ be a finite poset, with $|P_0| = \{x_1, \ldots, x_{m_0}\}$, and S any subset of $|P_0|$. Suppose that in $L_+(P_0;\, m_1, \ldots, m_{m_0})$ we fix nonnegative values for all the m_i such that $x_i \notin S$. Then (regardless of the values so chosen) the resulting function of the remaining variables (the m_i with $x_i \in S$) is given by a polynomial if and only if S is a chain in P_0.*

Sketch of proof. We have just seen "only if"; I shall sketch the proof of "if".

Alongside the fixed values for the m_i with $x_i \notin S$, let us choose arbitrary values for the m_i with $x_i \in S$. Now let Q denote the lexicographic sum over $(|P_0| - S, \preccurlyeq_0)$ of the chains C_i $(x_i \in |P_0| - S)$, and C the lexicographic sum over $(S, \preccurlyeq_0)$ of the chains C_i $(x_i \in S)$. Thus, C is a chain of $\sum_{x_i \in S} m_i$ elements. To specify a linearization of $P_0 * (C_1, \ldots, C_{m_0})$, we can first specify a linearization Q' of Q, then specify how the elements of the chain C are to be inserted among those of Q'. To complete the proof of the theorem, it will suffice to show that for each of the finitely many linearizations Q' of Q, the number of ways of positioning the members of C among the members of Q' is a polynomial in the variables m_i $(x_i \in S)$.

Given Q', we shall first show that there is a polynomial which gives that number whenever *positive* integer values are assigned to the m_i $(i \in S)$, then show that a polynomial with this property must continue to give that linearization-count when its arguments are allowed to be zero.

The complication in counting ways that members of C can be positioned is that the regions of Q' where the elements of the various subchains C_i of C can be inserted are possibly overlapping intervals, so that information on where members of one of those subchains are distributed within its allowed interval may or may not restrict where other subchains $C_{i'}$ can be distributed within theirs.

But suppose now that the m_i with $x_i \in S$ are all nonzero, so that the C_i are all nonempty. In this case, let us further classify linearizations of $Q' \cup C$ according to the positions, relative to the elements of Q', of the *greatest* elements of each of the chains C_i $(x_i \in S)$; i.e., according to which successive pair of the finitely many elements of Q' each of these greatest elements lies between, or whether it lies above or below all of Q'. Once such a set of positions has been specified, we know the linear ordering of the union of Q' with the set of maximal elements of those chains (since S is linearly ordered, hence so is the set of maximal elements of the C_i with $m_i \in S$). Let us call that union, so ordered, Q''. Then the intervals of Q'' in which the *remaining* elements of each C_i can be inserted are disjoint. If, for a given C_i, that interval of Q'' contains d_i elements (where d_i may be 0 if, for instance, the top members of C_i and of the next lower $C_{i'}$ have been placed between the same pair of elements of Q'), then there are $\binom{m_i-1+d_i}{d_i}$ ways to populate it with the $m_i - 1$ nonmaximal members of C_i. This is a polynomial in m_i (of degree d_i), hence the total number of linearizations of $P_0 * (C_1, \ldots, C_{m_0})$ extending our linearization of Q'' is

$$\prod_{x_i \in S} \binom{m_i-1+d_i}{d_i}, \tag{15}$$

which is a polynomial in the variables m_i $(x_i \in S)$. Summing over the finitely many linearizations Q' of Q, and the finitely many ways of positioning within each Q' the highest members of the chains C_i, we get a polynomial f which yields the value of $L_+(P_0;\, m_1, \ldots, m_{m_0})$ – provided that, as assumed above, none of m_i with $x_i \in S$ is zero.

The key to proving that the same polynomial works if one or more of those m_i is zero is Proposition 12(a), which says that if we fix all but one of the m_i, then $L_+(P_0;\, m_1, \ldots, m_{m_0})$ is a polynomial in that variable. Hence, if we choose for all of our m_i other than some particular m_{i_0} nonzero values, then the resulting polynomial function in the variable m_{i_0} must agree with the polynomial obtained in the preceding paragraph at all values of m_{i_0} except possibly 0. But two 1-variable real polynomials which agree at infinitely inputs are equal, so in fact they will also agree when $m_{i_0} = 0$. This shows that the result of the preceding paragraph

extends to the case where *at most one* of the m_i is zero. An obvious induction on the number of zero arguments gives the general case. □

The easy part of the above result, saying that if S is *not* a chain, then $L_+(P_0; m_1, \dots, m_{m_0})$ does *not* yield polynomial functions of the corresponding set of variables, does not have an obvious analog for the sign-imbalance function $L_-(P_0; m_1, \dots, m_{m_0})$, since it is based on finding a summand that is too big to be a polynomial; but in the computation of L_-, "big" summands of opposite sign can cancel. And indeed, we have seen classes of cases where the resulting function is 0, which is certainly a polynomial.

However, the hard part of the above theorem, concerning the case where S is a chain, does go over to L_-, mutatis mutandis:

Corollary 14 (to proof of Theorem 13)**.** *Let $P_0 = (|P_0|, \preccurlyeq_0)$ be as in Theorem 13, and let $S \subseteq |P_0|$ be a chain. Then if we fix nonnegative values for all the m_i with $x_i \notin S$, the values of $L_-(P_0; m_1, \dots, m_{m_0})$ are given by $2^{\mathrm{card}(S)}$ polynomials in the m_i such that $x_i \in S$, one for each choice of the parities of these* $\mathrm{card}(S)$ *variables.*

Idea of proof. Mimic the argument in the proof of Theorem 13 using, in place of the polynomials giving binomial coefficients as functions of one of their variables, the pairs of polynomials similarly describing the sign-imbalance of a union of two chains, illustrated by the right-hand side of (13) and formalized in Proposition 12(b). □

5.4. **The total degrees of our multivariable polynomials.** Proposition 12(a) described the degree of the one-variable polynomial it referred to. We can similarly determine the total degree of the $\mathrm{card}(S)$-variable polynomial of Theorem 13. In doing so, we will use the following curious lemma, which says that one can strengthen the ordering of a poset so as to get rid of one class of incomparability conditions, while preserving "enough of" another related class of such conditions.

Lemma 15. *Let $P = (|P|, \preccurlyeq)$ be a finite poset, and S a chain in P. Then the ordering $\preccurlyeq$ of P can be strengthened to an ordering $\preccurlyeq'$ under which the complement of S in P also becomes a chain, while every element of P that is incomparable with at least one element of S under $\preccurlyeq$ remains incomparable with at least one element of S under $\preccurlyeq'$.*

Proof. Given any pair of incomparable elements $x, y \in |P| - S$, consider the strengthenings $\preccurlyeq_{x,y}$ and $\preccurlyeq_{y,x}$ of $\preccurlyeq$ obtained by imposing the relation $x \preccurlyeq_{x,y} y$, respectively $y \preccurlyeq_{y,x} x$. We shall show that

(16) At least one of $\preccurlyeq_{x,y}$, $\preccurlyeq_{y,x}$ has the property that every element of P that is incomparable under $\preccurlyeq$ with at least one element of S remains incomparable under that strengthened ordering with at least one element of S.

Repeatedly strengthening our partial order in this way, we eventually get an ordering under which $|P| - S$ has no incomparable elements, i.e., is a chain, as desired.

It is easy to check (and probably well-known – reference, anyone?) that the relation $\preccurlyeq_{x,y}$ can be characterized by the condition that for all $w, z \in |P|$,

(17) $w \preccurlyeq_{x,y} z$ if and only if either $w \preccurlyeq z$, or $w \preccurlyeq x$ and $y \preccurlyeq z$.

Thus, if an element $u \in |P|$ satisfies a relation under $\preccurlyeq_{x,y}$ that it does *not* satisfy under $\preccurlyeq$, we must have either $u \preccurlyeq x$ or $y \preccurlyeq u$. (Note, incidentally, that these two cases are mutually exclusive, since x and y were assumed $\preccurlyeq$-incomparable.)

Suppose now that we have an element $u \in |P|$ such that

(18) u is $\preccurlyeq$-incomparable with at least one element of S, but is $\preccurlyeq_{x,y}$-comparable with all elements of S.

Of the two alternatives noted at the end of the sentence following (17), let us begin by assuming

(19) $u \preccurlyeq x$.

Let us write $\downarrow u$ for $\{z \in |P| \mid z \preccurlyeq u\}$ and $\downarrow_{x,y} u$ for $\{z \in |P| \mid z \preccurlyeq_{x,y} u\}$, and make the obvious corresponding definitions of $\uparrow u$ and $\uparrow_{x,y} u$. The statement that u is $\preccurlyeq_{x,y}$-comparable with all elements of S thus says that

(20) $(\downarrow_{x,y} u \cap S) \cup (\uparrow_{x,y} u \cap S) \;=\; S.$

Now from (17) and (19), it follows that

(21) $\quad \downarrow_{x,y} u = \downarrow u \quad$ and $\quad \uparrow_{x,y} u \;=\; \uparrow u \cup \uparrow y.$

(The first relation is gotten by putting u in the role of z in (17), and letting w range over P; the second by putting u in the role of w and letting z range over P.) Since S is a chain, $(\uparrow u \cup \uparrow y) \cap S$ must be either $\uparrow u \cap S$ or $\uparrow y \cap S$. If it were $\uparrow u \cap S$, then in view of (21), (20) would say that u was $\preccurlyeq$-comparable with all elements of S, contrary to the first condition of (18). Hence it is $\uparrow y \cap S$, and (20) instead says

(22) $\quad (\downarrow u \cap S) \cup (\uparrow y \cap S) \;=\; S.$

In view of (19), $\downarrow u \subseteq \downarrow x$, so (22) implies

(23) $\quad (\downarrow x \cap S) \cup (\uparrow y \cap S) \;=\; S.$

If, rather than (19) we are in the other case, $y \preccurlyeq u$, we get the variant of (23) with $\uparrow$ and $\downarrow$ reversed and the roles of x and y interchanged – which is again (23). So (23) holds in either case.

Now suppose that in addition to an element u satisfying (18), there is also an element $v \in |P|$ such that

(24) $\quad v$ is $\preccurlyeq$-incomparable with at least one element of S, but is $\preccurlyeq_{y,x}$-comparable with all elements of S.

Then we get the variant of (23) with only the roles of x and y interchanged:

(25) $\quad (\downarrow y \cap S) \cup (\uparrow x \cap S) \;=\; S.$

From (23) and (25), it is not hard to deduce that $\downarrow x \cap S = \downarrow y \cap S$ and $\uparrow x \cap S = \uparrow y \cap S$, and that these are complementary subsets of S; i.e., that both x and y are $\preccurlyeq$-comparable with all elements of S, and that the order relation of each element of S with x is the same as its order relation with y. (Quick Venn diagram proof: Draw a square, to represent properties of an element $s \in S$. Divide it by vertical lines into three regions according to the $\preccurlyeq$-relation of s to x: "smaller", "incomparable", or "greater"; and similarly by horizontal lines according to its $\preccurlyeq$-relation to y. Interpret each of (23) and (25) as saying that every $s \in S$ lies in a certain region of this diagram. Shade those regions, and note their intersection.)

It follows from this and (17) that passing from $\preccurlyeq$ to $\preccurlyeq_{x,y}$ does not affect the order-relation or lack of it between any element of P and any element of S (and similarly for $\preccurlyeq_{y,x}$). This contradicts our assumption (18) (and also (24)). Thus, our assumption that there existed both u satisfying (18) and v satisfying (24) has led to a contradiction, proving (16), and completing the proof of the lemma. □

Some further observations related to the above lemma are noted in an appendix, section 7.

We can now get the result we are aiming for. Recall that the *total degree* of a polynomial f in several variables means the maximum, over the monomials occurring in f, of the sum of the exponents of the variables.

Corollary 16 (to proof of Theorem 13)**.** *Let P_0 be a finite poset and S a chain in P_0, and, as in Theorem 13, consider the function $L_+(P_0; m_1, \ldots, m_{m_0})$ with fixed values chosen for all the m_i with $x_i \notin S$, as a function of the values of m_i with $x_i \in S$. Let f be the polynomial in the latter* $\mathrm{card}(S)$ *variables which, by that theorem, gives this function.*

Then the total degree of f is equal to the sum of m_i over those x_i that are incomparable in P_0 with at least one element of S.

Sketch of proof. The degree we are looking for will be the maximum of the total degrees, in the variables corresponding to the elements of S, of the *leading terms* of the polynomials (15) that are summed to get f. (A general multivariable polynomial does not have a well-defined "leading term", but the meaning of that phrase for the polynomials (15) is clear. When we sum the terms (15), the leading terms of some of these polynomials may be cancelled by negative-coefficient non-leading terms of higher-degree polynomials; but this cannot happen to any of the leading terms that have maximum total degree.) Now for each such polynomial, its total degree in the variables we are interested in is the sum over $x_i \in S$ of the number called d_i in the proof of Theorem 13; i.e., the number of elements of Q lying in the range into which the non-maximal members of the subchain C_i can be placed. Every element in one of those ranges must belong to a C_i $(x_i \in |P| - S)$ such that x_i is incomparable with at least one member of S; so the sum of the cardinalities of those chains C_i, i.e., the sum of the corresponding m_i, is indeed an upper bound for the desired degree.

To get a linearization Q' of the Q of that proof with the help of which we can realize that upper bound, we apply Lemma 15, taking for the P of that lemma the lexicographic sum over P_0 having a chain of length m_i in the i-th position for all i with $x_i \notin S$, while for $x_i \in S$, it has a singleton, which we continue to denote x_i. Taking the set of these singletons for the S of that lemma, we get an ordering $\preccurlyeq'$ of P that makes $|P| - S = |Q|$ a chain, and we take this linearization of Q to be our Q'.

To single out a polynomial (15), we now need to choose for each i with $x_i \in S$ the position where, in the construction of Theorem 13, we will locate the largest element of the chain C_i relative to the elements of Q'. Let us place each of these as high as we can consistent with the order $\preccurlyeq'$. I.e., for the top $x_i \in S$, we place x_{i,m_i} just below the least element of Q' that, under the ordering constructed, is above x_i, if there is one; if not, we place it above all elements of Q'; and we then place the top elements of successively lower chains C_i for $x_i \in S$ as high as they can go relative to Q' and the elements we have put down so far. One finds that every element of Q' incomparable under $\preccurlyeq'$ with at least one member of S is in the range which, in the construction of Theorem 13, can be populated by elements of C_i for some $x_i \in S$; so the asserted total degree is achieved. □

6. Associativity of the lexicographic sum, and its consequences

Suppose, as in our general description of lexicographic sums, that $P_0 = (|P_0|, \preccurlyeq_0)$ is a poset with $|P_0| = \{x_1, \dots, x_{m_0}\}$, and that for each $i \in \{1, \dots, m_0\}$ we are given a poset $P_i = (|P_i|, \preccurlyeq_i)$ with $|P_i| = \{x_{i,1}, \dots, x_{i,m_i}\}$, where $x_{i,j}$ and $x_{i',j'}$ are distinct unless $(i,j) = (i',j')$.

Now suppose further that for each pair (i,j) with $1 \le i \le m_0$ and $1 \le j \le m_i$, we are given a poset $P_{i,j} = (|P_{i,j}|, \preccurlyeq_{i,j})$ with $|P_{i,j}| = \{x_{i,j,1}, \dots, x_{i,j,m_{i,j}}\}$, such that $x_{i,j,k}$ and $x_{i',j',k'}$ are distinct unless $(i,j,k) = (i',j',k')$. Then we can define a partial ordering on

$$|P| \;=\; \{x_{i,j,k} \mid 1 \le i \le m_0,\ 1 \le j \le m_i,\ 1 \le k \le m_{i,j}\} \tag{26}$$

by letting

$$\begin{aligned} &x_{i,j,k} \preccurlyeq x_{i',j',k'} \text{ if and only if either} \\ &\quad i \ne i' \text{ and } x_i \preccurlyeq_0 x_{i'} \text{ in } P_0, \text{ or} \\ &\quad i = i' \text{ but } j \ne j', \text{ and } x_{i,j} \preccurlyeq_i x_{i,j'} \text{ in } P_i, \text{ or} \\ &\quad i = i' \text{ and } j = j', \text{ and } x_{i,j,k} \preccurlyeq_{i,j} x_{i,j,k'} \text{ in } P_{i,j}. \end{aligned} \tag{27}$$

Clearly, the resulting poset P can be looked at both as the lexicographic sum over P_0 of the posets $P_i * (P_{i,1}, \dots, P_{i,m_i})$ and as the lexicographic sum over $P_0 * (P_1, \dots, P_{m_0})$ of the posets $P_{i,j}$. Denoting their common value $P_0 * (P_i)_{1 \le i \le m_0} * (P_{i,j})_{1 \le i \le m_0,\, 1 \le j \le m_i}$, we thus have

$$\begin{aligned} &P_0 * ((P_i) * (P_{i,j})_{1 \le j \le m_i})_{1 \le i \le m_0} \\ &\qquad = \; P_0 * (P_i)_{1 \le i \le m_0} * (P_{i,j})_{1 \le i \le m_0,\, 1 \le j \le m_i} \\ &\qquad\qquad = \; (P_0 * (P_i)_{1 \le i \le m_0}) * (P_{i,j})_{1 \le i \le m_0,\, 1 \le j \le m_i}. \end{aligned} \tag{28}$$

The equality between the first and last lines of (28) constitutes an associative law for lexicographic sums. (Of course, if one is given families of posets essentially as above, but without the disjointness assumptions on their underlying sets, one can construct lexicographic sums using ordered tuples, as at the beginning of section 4.1, and one gets natural *isomorphisms*, rather than equalities, in (28).)

Now let us suppose each of the posets $P_{i,j}$ is a chain $C_{i,j}$ of $m_{i,j}$ elements, but make no such assumption on P_0 or the P_i. Then if we apply the function $L_\pm$ to the expressions in (28), the final expression is the function $L_\pm(P_0 * (P_i)_{1 \le i \le m_0}; m_{1,1}, \dots, m_{m_0,m_{m_0}})$, while the initial expression can be computed using Theorem 7 from the functions $L_\pm(P_0; m_1, \dots, m_{m_0})$ and $L_\pm(P_i; m_{i,1}, \dots, m_{i,m_i})$ $(i = 1, \dots, m_0)$.

Thus, if we know the functions $L_\pm(P; m_1, \dots)$ for some family of finite posets P, we can compute using Theorem 7 the corresponding function for any poset constructed as a lexicographic sum of members of that family over a member of that family; and, more generally, for any poset obtained in that way by iterated lexicographic sums.

In particular, since Propositions 9 and 10 give formulas for $L_\pm(P; m_1, \dots)$ when P is a finite chain or antichain, we can use the above technique to get such formulas for all posets constructed from chains and antichains by iterated lexicographic sums.

For example, consider the poset $P =$. If we name the bottom two elements $x_{1,1}$ and $x_{1,2}$, and the top two $x_{2,1}$ and $x_{2,2}$, then our poset is the lexicographic sum of two antichains with underlying sets $\{x_{1,1}, x_{1,2}\}$ and $\{x_{2,1}, x_{2,2}\}$, over a chain whose element-set we may label $\{x_1, x_2\}$. With the help of Propositions 9 and 10, we find that $L_+(P;\, m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2}) = \binom{m_{1,1}+m_{1,2}}{m_{1,1}}\binom{m_{2,1}+m_{2,2}}{m_{2,1}}$, while $L_-(P;\, m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2})$ is zero if either both $m_{1,1}$ and $m_{1,2}$ are odd, or both $m_{2,1}$ and $m_{2,2}$ are odd, while it is $\binom{\lfloor (m_{1,1}+m_{1,2})/2 \rfloor}{\lfloor m_{1,1}/2 \rfloor}\binom{\lfloor (m_{2,1}+m_{2,2})/2 \rfloor}{\lfloor m_{2,1}/2 \rfloor}$ otherwise. Using iterated lexicographic sums, one can build up from chains and antichains arbitrarily complicated posets for which these functions can similarly be computed. These are called "series-parallel" posets in [2, Chapter 9, Exercise 6].

But "most" finite posets are not series-parallel; the simplest example is $P =$. In fact, it is shown in [2, Chapter 9, Exercises 6–7] that a finite poset is series-parallel if and only if it does not contain a copy of that 4-element poset. For further results on the characterization of classes of posets arising as iterated lexicographic sums in terms of "forbidden subposets", see [3].

As mentioned at the end of section 4, I have not studied the function $L_\pm(P;\, m_1, m_2, m_3, m_4)$ determined by the above 4-element poset . It would be interesting to describe it.

7. Appendix: notes on Lemma 15

I don't know whether Lemma 15 has uses other than as a tool for proving Corollary 16; but it has piqued my curiosity, and I give below several related observations.

First, some quick examples. For a case of (16) in which one, but not the other of $\preccurlyeq_{x,y}$ and $\preccurlyeq_{y,x}$ has the property asserted there, let P be the 4-element poset , let S be the 2-element chain in the middle of that picture, and let x and y be (necessarily) the two elements of $|P| - S$. The reader can easily check the details.

For a case of the lemma in which the set of elements of S incomparable with one or more elements of P must always decrease when $|P| - S$ is made a chain (in contrast to the assertion of the lemma, about elements of P incomparable with one or more elements of S), let P be the disconnected union of a 3-element chain and a singleton, and take for S the subchain consisting of the top and bottom elements of the 3-element component.

Finally, for an example showing that the statement of the lemma fails if we drop the assumption that S is a chain, let P be the disconnected union of two 2-element chains, and let S consist of the two minimal elements of P. Then under any strengthening of the ordering of P that makes $|P| - S$ a chain, the larger element of $|P| - S$ will lie above, and in particular, be comparable with, both elements of S, though it was incomparable with one of them under the original ordering.

It is not obvious from the proof we gave of Lemma 15 how to tell, given elements x and y, which of the orderings $\preccurlyeq_{x,y}$ and $\preccurlyeq_{y,x}$ has the property asserted in (16), or whether both do. The equivalence (a) $\Longleftrightarrow$ (c) of the next corollary, and its variant with the roles of x and y reversed, give the criteria for one or the other of those possibilities to be excluded.

Corollary 17 (to proof of Lemma 15)**.** *Let $P = (|P|, \preccurlyeq)$ be a finite poset and S a chain in P, let x and y be two $\preccurlyeq$-incomparable elements of $|P| - S$, and let $\preccurlyeq_{x,y}$ be the strengthened partial ordering on P gotten by imposing the relation $x \preccurlyeq_{x,y} y$, described in* (17)*. Then the following three conditions are equivalent.*

(a) *There is some $u \in |P|$ that is incomparable under $\preccurlyeq$ with at least one element of S, but is comparable under $\preccurlyeq_{x,y}$ with every element of S.*

(b) *One of x or y has the above property of being incomparable under $\preccurlyeq$ with at least one element of S, but comparable under $\preccurlyeq_{x,y}$ with every element of S.*

(c) $(\downarrow x \cap S) \cup (\uparrow y \cap S) = S$, *but* $(\downarrow y \cap S) \cup (\uparrow x \cap S) \neq S$. (*Cf.* (23), (25)).

Proof. We shall prove (a) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a).

Given (a), the proof of Lemma 15 gives us the equality (23) with which (c) begins. On the other hand, if alongside (23) we have equality in place of the inequality in the second condition of (c) (i.e., if (25) also holds), then the two paragraphs of that proof following (25) show that $\preccurlyeq_{x,y}$ does not introduce any relations between elements of $|P| - S$ and elements of S that don't hold under $\preccurlyeq$, contradicting (a). So we must also have that inequality; so (c) indeed holds.

Next, assume (c). Note that in the equality $(\downarrow x \cap S) \cup (\uparrow y \cap S) = S$, the sets $\downarrow x \cap S$ and $\uparrow y \cap S$ must be disjoint, since if their intersection contained an element z, we would have $y \preccurlyeq z \preccurlyeq x$, contradicting our assumption that x and y are $\preccurlyeq$-incomparable.

Let us use the inequality $(\downarrow y \cap S) \cup (\uparrow x \cap S) \neq S$ to choose a $w \in S$ in neither $\downarrow y$ nor $\uparrow x$. By the preceding observation, w lies either in $\downarrow x$ or in $\uparrow y$, but not in both. If it is in $\downarrow x$ but not $\uparrow y$, then it is in neither $\uparrow y$ nor $\downarrow y$, showing that y is incomparable with at least one element of S. Now every $s \in S$ incomparable with y must, by the relation $(\downarrow x \cap S) \cup (\uparrow y \cap S) = S$, lie in $\downarrow x$, hence $s \preccurlyeq_{x,y} x \preccurlyeq_{x,y} y$. So y, though $\preccurlyeq$-incomparable with some elements $s \in S$, is $\preccurlyeq_{x,y}$-comparable with all such elements, which is the "y" case of (b). If, on the other hand, our element w lies in $\uparrow y$ rather than $\downarrow x$, the corresponding considerations give the "x" case of (b). Thus, we have proved (c) $\Longrightarrow$ (b).

The implication (b) $\Longrightarrow$ (a) is immediate. □

The proof of Lemma 15 shows how to build up *all* linearizations of $|P| - S$ which extend to orderings of $|P|$ of the sort asserted in the lemma, by making successive choices of order on unordered pairs of elements of $|P| - S$, taken in any order. The above corollary tells us at each such step which choices are available (those *not* satisfying (c)). We end this section with a more systematic construction of *some* ordering as in that lemma, based on a suggestion of Stefan Felsner (personal correspondence).

Sketch of an alternative proof of Lemma 15, after S. Felsner. Listing the elements of S as $s_1 \preccurlyeq \ldots \preccurlyeq s_r$, let us partition $|P| - S$ into disjoint subsets

$$|P| - S \;=\; T_1 \;\cup\; T_2 \;\cup\; \ldots \;\cup\; T_{2r+1} \tag{29}$$

as follows. If $x \in |P| - S$ is already $\preccurlyeq$-comparable with all elements of S, say with $s_i \preccurlyeq x \preccurlyeq s_{i+1}$, we assign x to T_{2i+1}, with the obvious modifications in the end-cases, namely, when $x \preccurlyeq s_1$ we assign it to T_1, and when $s_r \preccurlyeq x$ we assign x to T_{2r+1}. On the other hand, if x is incomparable with at least one element of S, let s_i be the largest such element, and assign x to T_{2i}.

It is not hard to check that for $1 \leq i < j \leq 2r+1$, no element of T_j is $\preccurlyeq$ any element of T_i. Hence we can strengthen the ordering $\preccurlyeq$ on $|P| - S$ to make all elements of T_i precede all elements of T_j whenever $i < j$, keeping the relative order of elements within each T_i. We can then go further and linearize each T_i, getting a total order $\preccurlyeq'$ on $|P| - S$.

On S, on the other hand, we let $\preccurlyeq'$ agree with $\preccurlyeq$, since $\preccurlyeq$ is already a total order there.

It remains to specify how $\preccurlyeq'$ should relate elements of $|P| - S$ and elements of S. If $x \in |P| - S$ belongs to a set T_{2i+1}, there is no choice: under $\preccurlyeq$, x lies above all s_j with $j \leq i$ and below all s_j with $j \geq i+1$, so we give it these same relations under $\preccurlyeq'$.

If $x \in T_{2i}$, we must again let x lie below all s_j with $j \geq i+1$. In this case, there may or may not be choices as to how it should relate to lower members of S; but we make a choice that will always work: let x be incomparable with s_i, and lie above all s_j with $j < i$.

It is routine, though tedious, to verify that the relation $\preccurlyeq'$ so defined is a partial ordering on $|P|$. By construction, it is a strengthening of the given ordering $\preccurlyeq$, and has the property that every element of $|P| - S$ that was $\preccurlyeq$-incomparable with at least one element of S (i.e., which belongs to some $|T_{2i}|$), remains $\preccurlyeq'$-incomparable with some element of S (namely, s_i). This completes the proof of the lemma. □

8. Appendix: a formula of G. Hochschild

The results of section 1 were motivated by a question Arthur Ogus asked me, on how one might understand, computationally, a formula of Gerhard Hochschild. In this appendix, which assumes only that section, we recover that formula.

Our development is far lengthier than Hochschild's, so its interest (if any) lies in its different approach to the result, and in the possibility that the method may be applicable to questions not as easy to answer by other means.

Hochschild's result (in which I have changed almost all the notation – but the translation between his and mine is straightforward) concerns an associative ring R of prime characteristic p, a commutative subring A of R, and an element $r \in R$ such that the commutator map $a \mapsto \mathrm{ad}_r(a) = ra - ar$ carries A into itself. What he shows is that for all $a \in A$,

$$(ar)^p \;=\; a^p r^p \;+\; \mathrm{ad}_{ar}^{p-1}(a)\, r. \tag{30}$$

Since every term of (30) ends with a factor r, Ogus suggested that the corresponding identity with those factors removed should hold, namely

$$(ar)^{p-1}a \;=\; a^p r^{p-1} + \mathrm{ad}_{ar}^{p-1}(a). \tag{31}$$

We shall see that this is true. Precisely, dropping the assumption that R has characteristic p, we shall prove

Lemma 18 (after Hochschild [1, Lemma 1])**.** *Let R be an associative ring, r an element of R, A a commutative subring of R such that the operation $\mathrm{ad}_r : a \mapsto ra - ar$ carries A into itself, and p a prime number. Then the function taking every $a \in A$ to the element*

$$(ar)^{p-1}a \;-\; a^p r^{p-1} \;-\; \mathrm{ad}_{ar}^{p-1}(a) \tag{32}$$

of R can be written as a noncommutative polynomial in r and the elements a, $\mathrm{ad}_r(a)$, $\mathrm{ad}_r^2(a)$, $\ldots$, in which the coefficients of all monomials are divisible by p.

Proof. Let us consider the term $(ar)^{p-1}a$ of (32), and repeatedly use the formula

$$rx \;=\; xr + \mathrm{ad}_r(x) \quad \text{for } x \in A \tag{33}$$

to eliminate occurrences of r preceding elements of A. We begin with the rightmost occurrence of r, which precedes the final a; an application of (33) to that pair of factors turns $(ar)^{p-1}a$ into a sum of two monomials, in one of which that r has jumped to the end, while in the other, it has been absorbed in the process of turning the final a to $\mathrm{ad}_r(a)$. We then apply (33) to the r that was originally second from the right. This can either be absorbed in the a immediately to its right, turning that into $\mathrm{ad}_r(a)$, or jump past it. In the latter case it can, in turn, either be absorbed in the next factor (which is a or $\mathrm{ad}_r(a)$ depending on which output of the first step we are looking at, and so is turned into $\mathrm{ad}_r(a)$ or $\mathrm{ad}_r^2(a)$ respectively), or jump past that factor, becoming an (additional) r at the far right. We proceed similarly with the third r from the right, and so on. Since there are two possibilities for the fate of the rightmost r, three for the next, etc., we get $p\,!$ terms.

The choices leading to one of these $p\,!$ terms can be represented visually by taking the given string

$$a\,r\,a\,r\,\ldots\,r\,a \qquad (\text{with } p\ a\text{'s and } p-1\ r\text{'s}) \tag{34}$$

and drawing, from each member of some subset of the occurrences of r, an arrow to some occurrence of a to its right. In the resulting expression, each a that receives m arrows becomes $\mathrm{ad}_r^m(a)$ (with those receiving none remaining as a), and the r's at the other ends of those arrows are deleted, while those r's from which arrows were not drawn move to the far right.

Note that the output of this process has exactly one term in which none of the $p-1$ r's acts on any of the a's, and that this term, $a^p r^{p-1}$, is cancelled by the $-a^p r^{p-1}$ of (32).

At the opposite extreme are the $(p-1)!$ terms in which *every* occurrence of r acts on an element of A (either an a, or the image of a under previous actions of other occurrences of r). I claim that the sum of these is precisely $\mathrm{ad}_{ar}^{p-1}(a)$, and thus cancels the $-\mathrm{ad}_{ar}^{p-1}(a)$ of (32). To see this, note first that for any $x \in A$, we have $\mathrm{ad}_{ar}(x) = arx - xar = arx - axr = a\,\mathrm{ad}_r(x)$, where the middle equality holds because A is commutative. Hence $\mathrm{ad}_{ar}^{p-1}(a)$ can be written as

$$(a\,\mathrm{ad}_r(\ldots(a\,\mathrm{ad}_r(a\,\mathrm{ad}_r(a)))\ldots))\,. \tag{35}$$

Here the rightmost (better: innermost) occurrence of ad_r acts on its argument a. The next occurrence acts on the product $a\ \mathrm{ad}_r(a)$, so – since ad_r is a derivation, i.e., satisfies

$$\mathrm{ad}_r(xy) \;=\; \mathrm{ad}_r(x)\,y + x\,\mathrm{ad}_r(y) \quad \text{for } x, y \in A, \tag{36}$$

– it turns $a\ \mathrm{ad}_r(a)$, into a sum of two terms, in one of which it acts on the first factor and in the other on the second. The action of the next ad_r, on the product of a with each of these two-factor terms, gives a sum of three terms; and so on. The resulting terms can be classified by writing out the string

$$a\ \mathrm{ad}_r\ \ldots\ a\ \mathrm{ad}_r\ a\ \mathrm{ad}_r\ a \qquad (\text{with } p\ a\text{'s and } p-1\ \mathrm{ad}_r\text{'s}) \tag{37}$$

and drawing an arrow from each ad_r to an arbitrary a to the right of it, on which it acts. The results are clearly the same as the subset of the expressions in our expansion of $(ar)^{p-1}a$ in which an arrow comes out of every occurrence of r, so, as claimed, these terms cancel the $-\mathrm{ad}_{ar}^{p-1}(a)$ in (32).

What remains is to show that in the expansion of $(ar)^{p-1}a$, each monomial

$$\mathrm{ad}_r^{m_1}(a)\ \mathrm{ad}_r^{m_2}(a)\ \mathrm{ad}_r^{m_p}(a)\ r^{p-1-m_1-\ldots-m_p} \tag{38}$$

that is *not* of either of the above two extreme sorts, i.e., which satisfies

$$0 \ < \ m_1 + \cdots + m_p \ < \ p-1, \tag{39}$$

occurs with coefficient divisible by p. Since A is commutative, we are regarding monomials (38) as the same if they differ by a permutation of the string of exponents $m_1, \ldots, m_p$. Thus, we may make the notational assumption that the *nonzero* exponents in (38) form an initial substring, say $m_1, \ldots, m_\ell$.

To determine the coefficient of a monomial (38), we need to count the ways of attaching arrows to (34) that lead to it. I claim such a system of arrows will be determined by an appropriately indexed family of elements of $\{1, \ldots, p\}$ (corresponding to the positions of the r's from which arrows begin, and the a's at which they end) subject to certain inequalities – and that the set of these indexed families can be identified with a set of maps of the sort whose cardinality was studied in section 1.

Indeed, consider any monomial (38), and let P be a partially ordered set consisting of ℓ connected components, namely, for each $i \leq \ell$, let the component P_i be a chain of $m_i + 1$ elements.

Let us, to begin with, assume for simplicity that the nonzero exponents in (38), $m_1, \ldots, m_\ell$, are distinct. Then for P so defined, let us, to each diagram of arrows on the string of symbols (34) that yields the monomial (38), associate the map $\varphi : P \to \{1, \ldots, p\}$ such that for $i = 1, \ldots, \ell$, if the factor $\mathrm{ad}_r^{m_i}(a)$ in (38) arises from arrows drawn from the k_1-st, k_2-nd, through k_{m_i}-th occurrences of r to the k_{m_i+1}-st occurrence of a, then φ maps the successive terms of the chain P_i to the integers $k_1 < k_2 < \cdots < k_{m_i} < k_{m_i+1}$. (Note the difference between the way $k_1, \ldots, k_{m_i}$ are specified in the above sentence, and way k_{m_i+1} is specified.)

Which maps $P \to \{1, \ldots, p\}$ can arise in this way from arrow-diagrams giving the monomial (38)? It is not hard to see that they will be precisely those *isotone* maps such that no two elements of P fall together, except that the *maximal* element of a component P_i is permitted to fall together with a *nonmaximal* element of a component P_j if $j \neq i$. (The images of non-maximal elements of P all have to be distinct because they represent the sources of distinct arrows, and the images of maximal elements must be distinct because they represent the recipients of distinct families of arrows. Finally, the images of a non-maximal element and the maximal element in the same component must be distinct because the i-th occurrence of r can't have an arrow to the i-th occurrence of a, since the latter precedes it. On the other hand, there is no contradiction if for some i, the i-th a is the recipient of some family of arrows, and the i-th r is the source of an arrow with a different destination. These are the cases where elements of P are allowed to fall together.) So letting $S = (|P|, \preccurlyeq, E)$, where $(|P|, \preccurlyeq)$ is the poset P described above, and E consists of all two-element subsets of $|P|$ other than those whose members are the maximal element of one component of P and a nonmaximal element of another component, we see that the coefficient of (38) in $(ar)^{p-1}a$ is $C(S, p)$, as defined in Theorem 1.

The partially ordered set P has $(m_1+1)+\cdots+(m_\ell+1) = m_1+\cdots+m_\ell+\ell$ elements, and ℓ connected components, so the term written $\mathrm{card}(|S|) - c + 1$ in Corollary 2 is here $m_1 + \cdots + m_\ell + 1$. By (39), $m_1 + \cdots + m_\ell < p - 1$, hence $m_1 + \cdots + m_\ell + 1 < p$, so p satisfies the condition in the first sentence of Corollary 2. By the first inequality of (39), P is nonempty, so we can apply the final statement of that corollary to conclude that $C(S, p)$ is divisible by p, as desired.

What if the m_i are not all distinct? If a given value m occurs as h_m different m_i's (i.e., if after collecting like factors in (38), $\mathrm{ad}_r^m(a)$ appears with exponent h_m), then the poset P constructed as above will have h_m m+1-element components. In this situation, for each way the h_m factors $\mathrm{ad}_r^m(a)$ can arise from an arrow-diagram, we can choose, arbitrarily, which of those h_m components of P is mapped to which family of m arrows. This gives $h_m!$ possibilities. We see from this that the coefficient of our monomial (38) in the expansion of $(ar)^{p-1}a$ will now be $C(S, p)/\prod_m h_m!$. But this creates no problem, since each of the h_m is less than p. (Indeed, by (39), $p - 1 > \sum m_i \geq h_m m \geq h_m$ for each m.) So since $C(S, p)$ is divisible by p, $C(S, p)/\prod_m h_m!$ is also, as required. □

9. Acknowledgements

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