Comments, corrections, and related references welcomed, as always!

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ADDENDA TO "ON COMMON DIVISORS OF MULTINOMIAL COEFFICIENTS"

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ABSTRACT. For k and N positive integers, let us understand a "proper k-nomial coefficient of weight N" to mean an integer $(n_1+\cdots+n_k)!/(n_1!\ldots n_k!)$ where all n_i are positive integers, and $n_1+\cdots+n_k=N$. Erdős and Szekeres [3] show that any two proper binomial coefficients of equal weight have a common divisor >1. The analogous statement for k k-nomial coefficients $(k>1)$ was conjectured in 1997 by David Wasserman.

In [1], after proving a conjecture of Erdős and Szekeres on the growth of the g.c.d. of two binomial coefficients of equal weight, I obtained some restrictions on possible counterexample to Wasserman's conjecture, mostly for $k=3$. That material was originally written up in a leisurely fashion. I subsequently rewrote it to "cut out the fat", and moved the deleted material, less central to that paper, to these pages, to which I later added one new result that I decided belonged in the same category.

I do not plan to publish this material.

In §1 we verify a computational result asserted without proof in [1], related to the the conjecture of Erdős and Szekeres proved there.

§2 below recalls some basic terminology from [1], and states Wasserman's conjecture.

In §3, to give a feel for the relevant considerations, we verify "by hand" that there is no counterexample to that conjecture with $k=3$ and $n=78$.

In §4, we strengthen examples from [1] showing that certain plausible generalizations of that conjecture fail, by proving that that, under a standard conjecture on prime values assumed by polynomials, there must be infinitely many such counterexamples, for all k.

In §5 we discuss how one of the methods of [1] might conceivably be strengthened.

In §6 we note how the proofs of two results in [1] can be modified to give estimates of different quantities from those estimated in the original results.

§1 can be read after [1, §2]; §2 reviews enough material so that it and §§3-4 can be read independent of [1]; §5 can be read after [1, §7]; §6 after [1, §6]. The sections of this note are independent of one another, except for the dependence of §§3-4 on the material from [1] recalled in §2.

1. Orbits of families of partitions.

I stated without proof in [1, §2] that given three orbits, under the natural action of S_N, of partitions of $\{1,\ldots,N\}$ into three subsets, the set of orbits into which the product of those orbits decomposes as an S_N-set formed a 20-parameter family. Let me sketch here what is involved, starting with general k in place of 3.

If a finite set of unspecified cardinality is given with k partitions into k subsets, then these together correspond to a decomposition into k^k subsets, so the resulting structure is described numerically by k^k natural numbers. If the cardinality of the whole set is then fixed at some value N, those natural numbers must sum to N, reducing the number of degrees of freedom by 1. (The set of cases is also reduced from infinite to finite; but we are interested in the number of integer parameters in a natural description of such a decomposition, regardless of the fact that the range of each parameter will be finite.) For each of the k decompositions into k subsets, we can specify next the cardinality of each of those subsets, by giving the first $k-1$ of these cardinalities (subject to the inequality saying that these $k-1$ values sum to $\leq N$), which determine the last one. Hence, specifying the cardinalities involved in each of these k decompositions reduces the number of degrees of freedom by $k(k-1)$.

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So given N, and the k expressions for N as a sum of k natural numbers, the number of degrees of freedom for the array of cardinalities of the intersections of our decompositions is $k^k - 1 - k(k-1)$; and each member of this array clearly corresponds to an orbit in the action of S_N on the set of k-tuples of decompositions. In particular, this formula gives values 1 and 20 respectively for $k = 2,\ 3$.

(Curious observation: each such $(k^k-1-k(k-1))$-parameter family of k^k-tuples of natural numbers will be the set of lattice points in some convex polytope of dimension $k^k - 1 - k(k-1)$, which, when each of the k partitions is into sets of equal cardinality and $k > 2$, should have a symmetry group of order at least $k!^{\,k+1}$.)

2. Background from [1]: Wasserman's conjecture, and some basic tools.

As in [1], for nonnegative integers $a_1, \dots, a_k$ we write

(1) $$\mathrm{ch}(a_1, \dots, a_k) \ = \ (a_1 + \dots + a_k)!\,/\,a_1!\ \dots\ a_k!\,,$$

and call (1) a *k-nomial coefficient* of *weight* $a_1 + \dots + a_k$. We call it a *proper* k-nomial coefficient if none of the a_i is zero. We also call a k-nomial coefficient a multinomial coefficient of *nomiality* k.

We are interested in

Conjecture 1 (David Wasserman, personal communication, 1997; cf. [4, p.131])**.** *For every $k > 1$, every family of k proper k-nomial coefficients of equal weight N has a common divisor > 1.*

Let us recall three easy identities for multinomial coefficients: first, a triviality statement for "monomial coefficients",

(2) $$\mathrm{ch}(a) \ = \ 1,$$

second, the "commutativity" of the operator ch, i.e., its invariance under permutation of its arguments,

(3) $$\mathrm{ch}(a_1, \dots, a_k) \ = \ \mathrm{ch}(a_{\pi(1)}, \dots, a_{\pi(k)}) \qquad \text{for } \pi \in S_k,$$

and finally, the associativity-like condition saying that for any string of strings of nonnegative integers, $a_1, \dots, a_{j_1};\ \dots\ ;\ a_{j_{k-1}+1}, \dots, a_{j_k}$, we have

(4) $$\mathrm{ch}(a_1, \dots, a_{j_k}) \\ = \ \mathrm{ch}(a_1 + \dots + a_{j_1},\ \dots\ , a_{j_{k-1}+1} + \dots + a_{j_k}) \cdot \mathrm{ch}(a_1, \dots, a_{j_1}) \cdot\ \dots\ \cdot \mathrm{ch}(a_{j_{k-1}+1}, \dots, a_{j_k}).$$

The next result, a key tool for studying the divisibility of properties of multinomial coefficients, was proved for binomial coefficients, i.e., for $k = 2$, by Kummer [6]; his proof extends without difficulty to general k.

Lemma 2 (after Kummer [6, 3rd from last display on p.116], cf. [5])**.** *Let $a_1, \dots, a_k$ be natural numbers, and p a prime. Then the power to which p divides $\mathrm{ch}(a_1, \dots, a_k)$ is equal to the number of "carries" that must be performed when the sum $a_1 + \dots + a_k$ is computed in base p.*

In particular, $\mathrm{ch}(a_1, \dots, a_k)$ is relatively prime to p if and only if that sum can be computed "without carrying", i.e., if and only if for each i, the sum of the coefficients of p^i in the base-p expressions for $a_1, \dots, a_k$ is less than p (and hence gives the coefficient of p^i in the base-p expression for $a_1 + \dots + a_k$). $\square$

As discussed in [1], one counts carries in a summation of several terms "in the obvious way"; but we don't need to review the details here, since we will only need the criterion of the last paragraph above.

As in [1], for N a positive integer and p a prime, we shall call a decomposition of N as a sum of positive integers $N = a_1 + \dots + a_k$ *p-acceptable* if $\mathrm{ch}(a_1, \dots, a_k)$ is not divisible by p; equivalently, if for each i, the i-th digit of the base-p expression for N is the sum of the i-th digits of the base-p expressions for $a_1, \dots, a_k$.

Thus, a failure of Conjecture 1 for given k and N would correspond to a family of k decompositions of N into k positive summands, such that for every prime p, at least one of these decompositions was p-acceptable.

3. A worked example

The results of [1] restricting possible counterexamples to Conjecture 1 for $k = 3$, and showing in particular that there are none with $N < 785$, were obtained after I had gotten some familiarity with the problem by hand calculations using the facts in the preceding section. Below, we show such calculations for one of the less trivial cases in the range $1 \leq N \leq 100$, namely $N = 78$.

Suppose we had a counterexample to the conjecture for that case. Starting with the greatest prime ≤ 78, namely 73, we note that our three decompositions must include one that is 73-acceptable. The base-73 expression for 78 is 15_{73}, from which it is easy to see that an expression $78 = a_1 + a_2 + a_3$ is 73-acceptable if and only if one of the a_i is ≥ 73. When this is so, $\mathrm{ch}(a_1, a_2, a_3)$ will be a divisor of $78!/73! = 78 \cdot 77 \cdot 76 \cdot 75 \cdot 74$, so our 73-acceptable decomposition of 78 is automatically p-acceptable for all primes p other than those dividing that product, i.e., other than

(5) $\quad 2,\ 3,\ 13;\ 7,\ 11;\ 19;\ 5;\ 37.$

Thus, a family of weight-78 trinomial coefficients will have no common divisor if and only if the decompositions of 78 on which it is based include a 73-acceptable decomposition, and include a p-acceptable decomposition for each p listed in (5).

Working out the expressions for 78 to each of the bases in question, we get

(6) $\quad 78 = 1001110_2 = 2220_3 = 60_{13} = 141_7 = 71_{11} = 42_{19} = 303_5 = 24_{37} = 15_{73}.$

The monotonically increasing values of the last digits are not a coincidence: We began the list (5) with the factors p of 78, for which the last digit of the base-p expression for 78 must be 0, followed these with the factors of 77, for which the last digit of the expression for 78 must be 1, and so on (omitting repeated primes). Looking at those final digits in terms of the criterion for p-acceptability (end of §2), we have the necessary (though not sufficient) condition

(7) Suppose $a_1 + a_2 + a_3$ is an expression for 78 as a sum of three positive integers, which is p-acceptable for a prime p. Then if $p = 2$, 3 or 13, all of a_1, a_2, a_3 are divisible by p; if $p = 7$ or 11, exactly two of them are divisible by p; while if $p = 19$, at least one of them is divisible by p.

Let us now show, for various pairs of primes p_1 and p_2 in the list (5), that an expression $78 = a_1 + a_2 + a_3$ cannot be both p_1- and p_2-acceptable. We begin by noting a consequence of the expression for 78 to base 2 shown in (6), together with an obvious general fact about 2-acceptable expressions.

(8) In any 2-acceptable expression for 78, at least one summand must be ≥ 64, and the powers of 2 dividing distinct summands must be distinct.

We claim that this implies

(9) No 2-acceptable expression for 78 as a sum of 3 positive integers can also be p-acceptable for any of the other primes p listed in (5), nor for $p = 73$.

We shall verify this for $p = 3,\ 13,\ 7,\ 11$ and 73, the cases we will actually need; the interested reader can easily verify it for the other cases. The expressions for 64 (cf. (8)) to these bases p are

(10) $\quad 64 = 2101_3 = 4\{12\}_{13} = 121_7 = 59_{11} = \{64\}_{73}$

(where I am using bracketed decimal expressions for digits larger than 9). Now consider an integer $a_1 \geq 64$ whose base-p expression has each digit less than or equal to the corresponding digit in the base-p expression for 78, and which leaves enough digits "free" to put together two other summands, a_2 and a_3, each of which is even (cf. (7)), and hence, when written to base p, has even digit-sum. Comparing (6) and (10), we see that there are no such possibilities for the first four values of p in (10) while for $p = 73$, the only such decomposition is $11_{73} + 2_{73} + 2_{73}$; but this fails to satisfy the condition in (8) that the powers to which 2 divides the summands be distinct. This establishes the cases of (9) that we shall use.

Let us look next at the prime 13; its relatively large size together with the condition from (7) that all summands be divisible by it constitute a strong combination.

(11) No 13-acceptable expression for 78 as a sum of 3 positive integers can also be p-acceptable for any of $p = 3,\ 7,\ 11$ or 73.

Indeed, for $p = 3$ we would have to have all summands divisible by both 13 and 3, hence each must be at least 39, which is impossible for more than two summands. Similarly, for $p = 7$ or 11, (7) says that *two* summands would have to be divisible by p, and since they would also be divisible by 13, this makes the sum far too large. Finally, for $p = 73$, one summand must be ≥ 73, so the remaining summands must add up to ≤ 5, making it impossible for them to be divisible by 13. (If one experiments further, one finds that a version of the argument used for $p = 7$ and 11 also works for $p = 19$, and a version of the argument for $p = 73$ works for $p = 37$. The corresponding statement is not true for $p = 5$: the decomposition $78 = 26 + 26 + 26$ is both 5- and 13-acceptable.)

We claim next that

(12) No expression for 78 as a sum of 3 positive integers can be p-acceptable for two values of p taken from among $7,\ 11,\ 73$.

Indeed, since any 7- or 11-acceptable expression for 78 must have two summands divisible by that prime, if an expression were both 7- and 11-acceptable, it would have to have at least one summand divisible by both primes, i.e., divisible by 77, hence the remaining two summands could add up to at most $78-77=1$, a contradiction. If an expression is 73-acceptable, then as one summand is at least 73, the other two can add up to at most 5, so it is impossible for all but one to be divisible by 7 or 11.

The results (9), (11) and (12) are more than enough to show that one cannot have a counterexample to Conjecture 1 with $N=78$: They tell us that any set of trinomial coefficients of weight 78 with no common divisor must have distinct members not divisible by the primes $p=2,\ 13,\ 7,\ 11$ and 73, hence must have not merely more than three, but more than four members. But let us push these arguments one more step, by showing that no decomposition of 78 that is p-acceptable for any p in this list of five primes can also be 3-acceptable.

We have already shown this for 2- and 13-acceptable decompositions, in (9) and (11) respectively. We have also observed that any 73-acceptable decomposition of 78 must have two summands adding up to ≤ 5, so they can't both be divisible by 3. Note next that $7\equiv 1 \pmod 3$, so any positive integer is congruent modulo 3 to the sum of its digits to base 7. But the digit-sum of 78 to base 7 is 6, so there is no 7-acceptable way of breaking 78 into more than two summands, each with base-7 digit-sum divisible by 3, as required by (7) for 3-acceptability. Finally, if an expression is both 3-acceptable and 11-acceptable, then two of its terms must be divisible by $3\cdot 11=30_{11}$, so each of these terms must equal 30_{11} or 60_{11}. The unique 11-acceptable expression for $78=71_{11}$ with two such summands is $30_{11}+30_{11}+11_{11}$; but to base 3 this expression has the form $1020_3+1020_3+110_3$, which is not 3-acceptable. So,

(13) The least k such that there exists a family of k proper trinomial coefficients of weight 78 without a common divisor is 6.

Of course, to assert this we must verify that there does exist such a six-element family. To get one, take $\mathrm{ch}(76,1,1)=78\cdot 77=2\cdot 3\cdot 13\cdot 7\cdot 11$, together with the five trinomial coefficients corresponding to p-acceptable decompositions of 78 for $p=2,\ 3,\ 13,\ 7$ and 11. (Using (6), one sees that there are several for each of these primes.)

4. Counterexamples to variant conjectures

Conjecture 1 says that a family of proper k-nomial coefficients of the same weight having no common divisor >1 must have at least $k+1$ members; but the result (13) of the preceding section suggests that it might be possible to increase this bound $k+1$. Let us prove the contrary. It is easy to give a particular counterexample for $k=3$; but we shall show that one can expect there to be infinitely many such examples for every $k>1$.

This will require us to find families of primes satisfying certain arithmetic relations. There is a standard conjecture on the distribution of primes, proposed by A. Schinzel [8], who called it "Hypothesis H", and extended by Bateman and Horn [2] to include an asymptotic estimate. In that form, it is known as the Bateman-Horn Conjecture, but we shall only need the original existence statement:

(14) *Schinzel's Conjecture* [8], cf. [7, p.11 ff.]. Suppose $f_1,\dots,f_k\in\mathbb{Z}[X]$ are irreducible polynomials with positive leading coefficients, and that for each prime p there exists at least one $r\in\mathbb{Z}/p\mathbb{Z}$ such that $f_1(r),\dots,f_k(r)$ are all nonzero. Then there exist infinitely many positive integers n such that $f_1(n),\dots,f_k(n)$ are all prime.

Let us first apply this to get examples for $k=3$. To do this, we will find integers N such that the set of prime factors of N and $N-1$ consists of $2,\ 3$ and two large primes p_1 and p_2, and such that some decomposition of N into three summands is simultaneously 2- and 3-acceptable. Then that decomposition, together with one p_1-acceptable decomposition, one p_2-acceptable decomposition, and the decomposition $(N-2)+1+1$, which handles all the other primes, will yield four trinomial coefficients of weight N with no common divisor. Here are the details.

Lemma 3. *There exists a positive integer N such that some family of four trinomial coefficients of weight N has no common divisor. Assuming Schinzel's Conjecture* (14)*, there are infinitely many such N.*

Proof. First note that for any positive integer of the form

(15) $\quad N \ = \ 2^4 \cdot 3^2 \cdot n \, + \, 15,$ where $n \equiv 0$ or $1 \pmod 3$,

the decomposition

(16) $\quad N \ = \ (N{-}15) + 12 + 3$

is both 2- and 3-acceptable. Indeed, to base 2, $N - 15$ has last four digits 0000_2 and the other two summands are 1100_2 and 11_2, giving 2-acceptability, while in base 3, we similarly have a summand ending $i00_3$ where i is 0 or 1, and summands 110_3 and 10_3, giving 3-acceptability. (We could shorten the wording above and below by restricting attention to one of the cases $n \equiv 0$ or $n \equiv 1$, but let us, for the moment, cast our nets as wide as possible.)

N will be divisible by 3, and $N{-}1$ by 2; let us write $N/3$ and $(N{-}1)/2$ as (linear) polynomials in n :

(17) $\quad f_1(n) \ = \ N/3 \ = \ 2^4 \cdot 3 \cdot n + 5, \qquad f_2(n) \ = \ (N-1)/2 \ = \ 2^3 \cdot 3^2 \cdot n + 7.$

In each of these polynomials, the coefficient of n has 2 and 3 as its only prime factors; hence modulo any prime $p \neq 2,\ 3,$ each of these polynomials has exactly one root among the p elements of $\mathbb{Z}/p\mathbb{Z}$. Hence these two polynomials are simultaneously nonzero at at least $p - 2 > 0$ residues $r \in \mathbb{Z}/p\mathbb{Z}$. Modulo 2 and 3, on the other hand, these polynomials are both nonzero constants. Hence the hypotheses of Schinzel's Conjecture (14) are satisfied, so assuming that conjecture, there exist infinitely many n such that $N/3$ and $(N{-}1)/2$ are prime; i.e., such that we can write $N = 3p_1$ and $N - 1 = 2p_2$ for some primes p_1 and p_2. Clearly, if we restrict attention to n of the form $3s$ or $3s + 1$ (as needed for 3-acceptability of (16)), the same argument applies to each of these subcases.

For $n,\ N,\ p_1,\ p_2$ so chosen, consider the four decompositions of N :

(18) $\quad (N{-}2) + 1 + 1, \qquad (N{-}15) + 12 + 3, \qquad p_1 + p_1 + p_1, \qquad p_2 + p_2 + 1.$

The trinomial coefficient corresponding to the first is $N(N-1) = 3{\cdot}p_1{\cdot}2{\cdot}p_2$, hence is not divisible by any prime but those four; we have seen that the second decomposition is 2- and 3-acceptable, and the last two are respectively p_1- and p_2-acceptable. Hence, as claimed, no prime divides all four trinomial coefficients.

For explicit examples, one finds that for $n = 1$ and $n = 3$, the values of (17) are indeed primes; the resulting values of N are 159 and 447. □

In our generalization of the above construction, we will want an integer M (corresponding to the summand 15 in (15)) which has a decomposition acceptable with respect to a certain list of primes, and which belongs to specified residue classes. The next lemma will allow us to find such an M.

Lemma 4. *Let $q_1, \dots, q_j$ be distinct primes, let $[r_1], \dots, [r_j]$ be residue classes modulo specified powers of these respective primes, and let h be a positive integer. Then there exists an integer $M \in [r_1] \cap \dots \cap [r_j]$ having a decomposition into h positive summands, $M = m_1 + \dots + m_h$, that is q_i-acceptable for $i = 1, \dots, j$.*

Proof. For $h = 1$ this is immediate: take any positive $m_1 \in [r_1] \cap \dots \cap [r_j]$, noting that by (2) the q_i-acceptability conditions are vacuous in the case of a single summand. Now let $h > 1$, and assume inductively that we have an $M' = m_1 + \dots + m_{h-1}$ satisfying the indicated condition for $h - 1$. For each $i = 1, \dots, j$, choose a power $q_i^{e_i}$ which is larger than M', and is also at least the power of q_i to which the residue $[r_i]$ is specified. Choose m_h to be any positive integer that is divisible by all the $q_i^{e_i}$, and let $M = M' + m_h$. Then the decomposition $M = m_1 + \dots + m_{h-1} + m_h$ has the desired properties: the q_i-acceptability conditions carry over because when we add m_h to $M' = m_1 + \dots + m_{h-1}$ in base q_i, all the digits of m_h below the q^{e_i} digit are zero, while all higher digits of M' are zero. □

Proposition 5. *Assuming Schinzel's Conjecture* (14)*, there will exist, for every integer $k > 1$, infinitely many positive integers N such that there exists a family of $k + 1$ proper k-nomial coefficients all of weight N and having no common divisor > 1.*

Proof. Given k, choose any distinct primes $q_1, \dots, q_{k-1}$ which include all primes $\leq k - 1$, and then choose positive integers $r_1, \dots, r_{k-1}$ such that $q_i^{r_i} \geq k - 1$. (In the proof of Lemma 3, the corresponding values were $k = 3,\ q_1 = 3,\ q_2 = 2,\ r_1 = r_2 = 1$.) Using the preceding lemma, let us choose a positive integer M with a decomposition

(19) $\quad M \ = \ m_1 + \dots + m_{k-1}$

which is q_i-acceptable for all i, and such that for each i, $M{-}(i{-}1)$ is divisible by $q_i^{r_i}$, but not by $q_i^{r_i+1}$. (These divisibility conditions can be gotten by specifying for each i that the residue of M modulo $q_i^{r_i+1}$

be $q_i^{r_i}+i-1$.) Note that since $q_i^{r_i}\geq k-1$, this insures that *none* of $M,\ M-1,\ldots,\ M-k+1$ is divisible by $q_i^{r_i+1}$.

Now choose for each $i=1,\ldots,k-1$ an integer $e_i>r_i$ such that $q_i^{e_i}>M$ and let $K=q_1^{e_1}\ldots q_{k-1}^{e_{k-1}}$. ($K$ plays the role of the $2^4\cdot 3^2$ of our previous lemma. There we did not quite have $3^2>15$, but we fudged that by restricting the residues mod 3 of the n that we allowed.) Consider the sequence of integers

$$N\ =\ M\ +\ Kn\quad (n\geq 1). \tag{20}$$

To show that infinitely many of these have the asserted property, let us, for $i=1,\ldots,k{-}1$, write Q_i for the greatest common divisor of $M-(i-1)$ and K. We claim that the polynomials

$$f_i(n)\ =\ (M-(i-1))/Q_i\ +\ (K/Q_i)\,n\qquad (i=1,\ldots,k-1) \tag{21}$$

satisfy the hypotheses of Schinzel's Conjecture. Indeed, for any prime p other than $q_1,\ldots,q_{k-1}$, the integer K, and hence each K/Q_i, is relatively prime to p; so each f_i has exactly one zero modulo p. Since $p>k-1$ (by choice of $q_1,\ldots,q_{k-1}$), $\mathbb{Z}/p\mathbb{Z}$ includes at least one residue which makes all $k-1$ of these polynomials nonzero. On the other hand, for $p=q_j$, we have seen that each $M-(i-1)$ is divisible by p to at most the power r_j, while K is divisible by that prime to at least the power r_j+1; hence by the choice of Q_i as the greatest common divisor of $M-(i-1)$ and K, $(M-(i-1))/Q_i$ is not divisible by p, and K/Q_i is; hence modulo p, each f_i is a nonzero constant. Hence Schinzel's Conjecture implies that for infinitely many n, all $f_i(n)$ will be primes p_i. Fixing such a value of n, let N be defined by (20). The condition defining $p_i=f_i(n)$ can be written as $Q_i\,p_i=M-(i-1)+Kn$, hence

$$N\ =\ M+Kn\ =\ Q_i\,p_i+(i-1)\qquad (i=1,\ \ldots,\ k-1). \tag{22}$$

Since our n and the p_i were chosen after Q_i, we may also insure by taking n large enough that

$$p_i\ >\ Q_i\quad \text{for all } i. \tag{23}$$

Now consider the following $k+1$ decompositions of $N=M+Kn$ into k summands. (The last $k{-}1$ of these decompositions come from the $k{-}1$ cases of (22).)

$$\begin{array}{lll}
(24) & (N-(k-1))+1+\cdots+1 & (\text{one term } N-(k-1), \text{ and } k-1 \text{ terms } 1),\\
& m_1+\cdots+m_{k-1}+Kn, & (\text{by (19)})\\
& p_1+\cdots+p_1+(Q_1-(k-1))p_1 & (k-1 \text{ terms } p_1, \text{ and one term } (Q_1-(k-1))p_1),\\
& p_2+\cdots+p_2+(Q_2-(k-2))p_2+1 & (k-2 \text{ terms } p_2, \text{ one term } (Q_2-(k-2))p_2, \text{ and one term } 1),\\
& \quad\ldots & \quad\ldots\\
& p_{k-1}+(Q_{k-1}-1)p_{k-1}+1+\ldots+1 & (\text{one term } p_{k-1}, \text{ one term } (Q_{k-1}-1)p_{k-1}, \text{ and } k-2 \text{ terms } 1).
\end{array}$$

The k-nomial coefficient corresponding to the first line of (24) equals $N!/(N-(k-1))!=N(N-1)\ldots(N-(k-2))=Q_1\,p_1\,\ldots\,Q_{k-1}\,p_{k-1}$, hence it is relatively prime to p for all primes p other than the p_i and the divisors of the Q_i, namely the q_i. The q_i are all taken care of by the k-nomial coefficient corresponding to the second line, by our choice of $m_1,\ldots,m_{k-1}$ and K. Finally, $p_1,\ldots,p_{k-1}$ are taken care of successively by the remaining $k-1$ decompositions. Indeed, since $Q_i\geq q_i^{e_i}>q_i^{r_i}\geq k-1$, the coefficients appearing in these decompositions are all positive, while (23) guarantees that when we add up Q_i terms p_i, we have no carrying in base p_i. $\square$

Though designed to give examples for large k, this lemma also shows us how to quickly generate examples for $k=2$. In particular, taking $q_1=2,\ M=2,\ K=4,\ n=1$, we see that $N=6$ is such an example (as can be discovered equally easily by perusal of Pascal's triangle).

An example with $k=3$ that does not arise as in the preceding lemmas was found by David Wasserman. Let $N=119$, and consider the four expressions

$$117+1+1,\qquad 59+59+1,\qquad 105+7+7,\qquad 68+34+17. \tag{25}$$

The last of these is both 2- and 17-acceptable; 2-acceptability is a result of the fact that 17 has "sparse" digits to base 2, and $119/17=7$ is small enough to "fit into the spaces between those digits". The other three decompositions take care of all other primes in the usual way. Gary Sivek (personal communication) has found by a computer check that 119 is the *second* smallest value of N admitting such a family of decompositions, the smallest being 95, for which one has

$$93+1+1,\qquad 80+10+5,\qquad 57+19+19,\qquad 1+47+47. \tag{26}$$

Here the second decomposition has properties analogous to those noted above for the last decomposition in (25). Sivek also notes that the third decomposition in each of (25) and (26) can be replaced by one or more variants. It is not evident whether Schinzel's Conjecture could be used to deduce that there should be infinitely many examples with properties like those of (25) and (26).

As noted in [1, 4th parag. of §9], another direction in which one might like to generalize Conjecture 1 is to allow families of proper multinomial coefficients of different nomialities, though still of equal weight. An appealing idea is that since we have conjectured that for k-nomial coefficients, any k (or fewer, of course) will have a common divisor, the condition guaranteeing a common divisor for coefficients of mixed nomialities should be that the sum of the *reciprocals* of the nomialities be ≤ 1.

However, this does not work. Indeed, if in the last $k-1$ lines of (24) we replace each of the terms $(Q_i-(k-i))p_i$ by $Q_i-(k-i)$ terms p_i, then p_i-acceptability continues to hold, but the nomiality increases to Q_i+i-1, so the sum of the reciprocals of the nomialities becomes $k^{-1}+k^{-1}+Q_1^{-1}+(Q_2+1)^{-1}+\cdots+(Q_{k-1}+(k-2))^{-1}$, which can be made arbitrarily small by taking k and each of the Q_i large. For a family of examples some of which involve numbers that are quite small, and which does not require Schinzel's conjecture, we note

Lemma 6. *There exist positive integers N for which one can find one proper binomial coefficient, and two proper multinomial coefficients of nomialities ≥ 4, all of weight N, which have no common divisor >1.*

Such examples can be found with the nomialities of the latter two terms arbitrarily high, hence with the sum of the reciprocals of the nomialities of the three terms arbitrarily close to $1/2$.

Proof. Let p_1 be any prime >3, let p_2 be any prime $>p_1$ which when written to base p_1 has digit-sum ≥ 4 (for instance, any prime $>p_1$ having remainder ≥ 3 modulo p_2), and let $N=p_1\,p_2$. For our binomial coefficient, take $\mathrm{ch}(N-1,\,1)=N$, which is divisible by no primes other than p_1 and p_2. Let our second multinomial coefficient correspond to the expression for N as a sum of p_1 repetitions of p_2; this expression is p_2-acceptable because $p_2>p_1$. Finally, let our last coefficient correspond to the expression for N to base p_1 (i.e., the expression for p_2 to base p_1, with a zero added at the end), with each summand $d_i\,p_1^i$ replaced by d_i repetitions of p_1^i; this is clearly p_1-acceptable. The nomialities of these last two coefficients are respectively p_1 and the sum of the digits of p_2 to base p_1, hence they can be made arbitrarily large, and under the assumptions we have made, are both ≥ 4, as required. □

The least such example, with $N=65$, is recorded in the indicated paragraph of [1].

Lest one conjecture some statement involving hypotheses on the nomialities other than their being large enough, note that from examples with given nomialities, one can get examples with any smaller nomialities, by collecting terms, as in (4), in an arbitrary way.

Although Proposition 5 can be thought of as weakly supporting Conjecture 1, by suggesting that it draws the line in the right place, I find the failure of the statement about sums of reciprocals troubling, since that seems such a natural extension of the conjecture. The fact that the conjecture requires $k>1$ is also worrisome.

Yet another plausible strengthening of Conjecture 1 is suggested, and demolished, in [1, 2nd parag. of §9].

5. How one might use primes dividing $N-h$ for $h>2$.

I commented in [1, 2nd parag. of §10] that to exclude a possible counterexample to Conjecture 1 in which the decomposition with the largest summand, say

$$(27)\qquad N \;=\; (N-i-j)+i+j$$

(i and j “small”) had $i+j>2$ (and hence, as proved there, $i+j\geq 11$), it might be necessary to make use, not only of the primes dividing $N,\,N-1$ and $N-2$, as was done in the results proved there, but also the primes dividing $N-3,\ldots,N-i-j+1$. The difficulty in doing so is that whereas it easily follows from the criterion for p-acceptability that a prime p dividing $N,\,N-1$ or $N-2$ must divide all three, respectively two, respectively at least one of the summands in a p-acceptable expression for N, this is not true for primes dividing $N-3,\ldots,N-i-j+1$.

To see how we might nonetheless make use of these primes, let us consider a prime p dividing $N-5$ (which is far enough from $N-2$ to make clear what is involved, but close enough not to get too messy), and such that the decomposition (27) is not p-acceptable, so that in a counterexample to Conjecture 1, one of the other decompositions in question, say $N=a_1+a_2+a_3$, must be p-acceptable. For simplicity of discussion, say $p^d > 5$. Since there is no carrying when $a_1 + a_2 + a_3$ is added in base p, there will, a fortiori, be none when they are added in base p^d; so, as the base-p^d representation of N ends in a 5, the remainders of a_1, a_2 and a_3 on division by p^d must be natural numbers summing to 5. Hence if none of these is 0, they must, up to a permutation, be either $3,\ 1,\ 1$ or $2,\ 2,\ 1$. So even though none of the a_i need be divisible by p^d, one of a finite list of linear combinations of these (such as $a_1 - 3\,a_2$) will be so divisible (though it may be zero). Perhaps such considerations can be used to extend the methods used in [1, Lemmas 7 and 10].

6. Variants of Proposition 7 and Lemma 9 of [1].

The calculations that were used in Proposition 7 and Lemma 9 of [1] to show that when $i = j = 1$, N must be large, while when $i + j > 2$, a certain integer C obtained from our decompositions of N must be large, can be modified to yield lower bounds for the a_i and b_i in these situations:

Lemma 7. *Suppose* (*as in* [1, Proposition 7]) *that N is a positive integer having decompositions with positive integer summands,*

$$(28) \qquad N \ = \ (N-2) \ + \ 1 \ + \ 1, \qquad N \ = \ a_1 \ + \ a_2 \ + \ a_3, \qquad N \ = \ b_1 \ + \ b_2 \ + \ b_3,$$

such that for every prime p, at least one of the decompositions of (28) *is p-acceptable.*

Suppose, moreover, that of these three decompositions, the second is 2*-acceptable.*

Then for $i = 1, 2, 3$ we have $a_i \geq 216 = 2^3 \cdot 3^3$ and $b_i \geq 256 = 2^8$, while if N is even, we in fact have $a_i \geq 864 = 2^5 \cdot 3^3$ and $b_i \geq 1024 = 2^{10}$.

Sketch of proof. We follow the proof of [1, Proposition 7] exactly up to display [1, (21)]. At that step, we either replace the estimate $a_1\,a_2\,a_3 \ \leq \ N^3/\,3^3$ used there with $a_1\,a_2\,a_3 \ \leq \ a_1\,N^2/\,2^2$, or the estimate $b_1\,b_2\,b_3 \ \leq \ N^3/\,3^3$ with $b_1\,b_2\,b_3 \ \leq \ b_1\,N^2/\,2^2$.

In the former case, noting that "divides" implies "$\leq$" in [1, (22)], and substituting the above modified version of [1, (21)] into the resulting inequality, we get

$$(29) \qquad 2\,N^3(N{-}1)^2 \ \leq \ a_1\,N^2/\,2^2 \cdot N^3/\,3^3, \quad \text{and if } N \text{ is even,} \quad 8\,N^3(N{-}1)^2 \ \leq \ a_1\,N^2/\,2^2 \cdot N^3/\,3^3.$$

Solving for a_1, we conclude that $a_1 \geq 2^3\,3^3\,(1{-}N^{-1})^2$ in general, and $\geq 2^5\,3^3\,(1{-}N^{-1})^2$ when N is even. Now $(1{-}N^{-1})^2 > 1 - 2N^{-1}$, and by the result of the original proposition, $N^{-1} \leq 1726^{-1}$ in general, and $\leq 6910^{-1}$ when N is even; so in either case, the product of N^{-1} and the numerical coefficient $2^3\,3^3$, respectively $2^5\,3^3$, is < 1, and so does not affect our inequality of integers. Thus we get the desired bound on a_1, and by symmetry, this applies equally to all the a_i.

If we use b_1 instead of a_1 in our modification of [1, (21)], we can also use the strengthening of the first inequality of [1, (21)] given by [1, (26)], namely $a_1\,a_2\,a_3 \ \leq \ N^3/\,2^5$. This gives

$$(30) \qquad 2\,N^3(N{-}1)^2 \ \leq \ N^3/\,2^5 \cdot b_1\,N^2/\,2^2, \quad \text{and if } N \text{ is even,} \quad 8\,N^3(N{-}1)^2 \ \leq \ N^3/\,2^5 \cdot b_1\,N^2/\,2^2.$$

Treating these similarly, we get the asserted lower bounds on the b_i. □

Remark: In the inequalities $a_1\,a_2\,a_3 \ \leq \ a_1\,N^2/\,2^2$ and $b_1\,b_2\,b_3 \ \leq \ b_1\,N^2/\,2^2$ used above, the right-hand sides can, of course, be lowered to $a_1\,(N{-}a_1)^2/\,2^2$ and $b_1\,(N{-}b_1)^2/\,2^2$. But the resulting improvements of our lower bounds become small for N large, so we have omitted them for simplicity.

Whereas the preceding result gave lower bounds on the a_i and b_i independent of N, and the result of [1, Lemma 9] gives a similar lower bound on C, the next result gives, in the case $i + j > 2$, a lower bound relating the a_i, the b_i and C which grows linearly in N. Let us first recall from [1, pp.7-8] some concepts that we will assume in the statement of the result, including the definition of C. (The reader might first read the first sentence of Lemma 8, and then the following definitions.)

(31) *Under the hypotheses* of the first sentence of Lemma 8 below –
A prime p is called *relevant* if it divides $N(N{-}1)(N{-}2)$, but if neither the second nor the third of the decompositions of (32) is p-acceptable.
A *relevant prime power* factor of N, $N{-}1$, or $N{-}2$ means the largest power of some relevant prime dividing the term in question.
The product of all relevant prime power factors of $N{-}2$, the squares of all relevant prime power factors of $N{-}1$, and the cubes of all relevant prime power factors of N, is denoted C.

Lemma 8. *Suppose* (*as in* [1, Lemma 9]) *that* N *is a positive integer having decompositions with positive integer summands,*

$$N = (N-i-j) + i + j, \qquad N = a_1 + a_2 + a_3, \qquad N = b_1 + b_2 + b_3, \tag{32}$$

such that $i+j>2$, *and for every prime* p, *at least one of the decompositions of* (32) *is* p*-acceptable.*

Then $C\cdot\min(a_1,\,a_2,\,a_3,\,b_1,\,b_2,\,b_3) \ \geq\ 108\,(N-4)$.

Further, if the second decomposition of (32) *is* 2*-acceptable* (*so that the prime* 2 *is not relevant, in the sense defined above*), *then* $C\cdot\min(a_1,\,a_2,\,a_3)\ \geq\ 216\,(N-4)$ *and* $C\cdot\min(b_1,\,b_2,\,b_3)\ \geq\ 256\,(N-4)$.

Proof. If 2 is a relevant prime, then using again the estimates $a_1\,a_2\,a_3 \leq a_1 N^2/2^2$ and $b_1\,b_2\,b_3 \leq N^3/3^3$, but substituting these into [1, (33)] rather than [1, (22)], we get

$$N^3(N-1)^2(N-2)\,/\,C \ \leq\ a_1\,N^2/\,2^2\,\cdot\,N^3/\,3^3, \tag{33}$$

hence

$$C\,\cdot\,a_1 \ \geq\ 2^2\,3^3\,N(1-N^{-1})^2(1-2N^{-1}). \tag{34}$$

As noted in [1, proof of Lemma 9], $(1-N^{-1})^2(1-2N^{-1}) \geq 1-4N^{-1}$, so multiplying out the right-hand side of (34), we have $C\cdot a_1 \geq 108\,(N-4)$. By symmetry, the same inequality holds for all a_i and b_i, giving the first conclusion of our result, under the assumption that 2 is a relevant prime.

Now the inequalities asserted by our lemma under the assumption that 2 is *not* a relevant prime are stronger than those asserted in general, so to complete our proof, it will suffice to prove those inequalities under that assumption. Since the assumption in the statement of the lemma as to which decomposition is 2-acceptable weakens our symmetry, we must estimate a_1 and b_1 separately.

The proof of [1, Lemma 9] shows that for 2 not a relevant prime, we have the analog of [1, (33)] with an extra factor of 2 on the left.

The estimate of a_1 now works exactly as before. With the extra factor of 2, we get $C\cdot a_1 \geq 216\,(N-4)$. Again, symmetry gives the corresponding inequalities for all a_i, yielding the first inequality of the final statement of the lemma.

On the other hand, when we put $b_1\,b_2\,b_3\ \leq b_1 N^2/\,2^2$ into [1, (33)], we can also, as in [1], use the fact that the decomposition $a_1+a_2+a_3$ is 2-acceptable to improve our other inequality to $a_1\,a_2\,a_3\ \leq N^3/\,2^5$, getting

$$2\,N^3(N-1)^2(N-2)\,/\,C \ \leq\ N^3/\,2^5\cdot b_1\,N^2/\,2^2, \tag{35}$$

and hence

$$C\,\cdot\,b_1 \ \geq\ 2^8\,N(1-N^{-1})^2(1-2N^{-1}), \tag{36}$$

which leads, as above, to the final inequality of our conclusion. □

One can play further variations on this technique, “leaving in” more than one of $a_1,\,\dots\,,\,b_3$ when we estimate $a_1\,a_2\,a_3\,b_1\,b_2\,b_3$. The relvant calculations should be clear. I do not know whether the resulting inequalities are likely to be useful.

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