Comments, corrections, and related references welcomed, as always!

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SOME EMPTY INVERSE LIMITS

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To the memory of Leon Henkin, 1921-2006

ABSTRACT. A simplified proof is given of L. Henkin's result that every directed partially ordered set I of uncountable cofinality indexes a system of nonempty sets and surjective set maps with empty inverse limit. Strengthening a result of N. Aronszajn, rediscovered by G. Higman and A. H. Stone, it is then shown that a large class of such partially ordered sets index systems with these properties in which the sets occurring are countable. Examples are also given of groups G admitting ω_1-indexed systems of transitive G-sets with empty inverse limits, whence their group rings $k\,G$ admit systems of cyclic modules with surjective homomorphisms and zero inverse limits. Several questions are posed.

1. Introduction, definitions, and overview.

Recall that a partially ordered set $(I, \leq)$ is said to be (upward) directed if

(1) for all $i, j \in I$, there exists $k \in I$ with $k \geq i$ and $k \geq j$,

and that an *inversely directed system* (or *inverse system*) of sets indexed by the directed partially ordered set I means a family (X_i, X_{ji}) such that

(2) for all $i \in I$, X_i is a set,

(3) for all $i \geq j$ in I, X_{ji} is a set map $X_i \to X_j$,

(4) for all $i \in I$, X_{ii} is the identity map of X_i,

(5) for all $i \geq j \geq k$ in I, one has $X_{ki} = X_{kj}\,X_{ji}$.

In this situation, the *inverse limit* of this system is defined to be the set

(6) $\varprojlim_{i\in I} X_i \ = \ \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid \forall\ i \geq j,\ X_{ji}(x_i) = x_j\}.$

(I generally prefer to reverse the inequalities in (1)-(6), and thus index inverse systems by *downward* directed sets, since the standard way to make a partially ordered set into a category is to have morphisms go from lower to higher elements, and it is preferable to have functors covariant when there is no intrinsic reason for the opposite choice. However, in most of the explicit examples in this note, this would require introducing the opposites of familiar partial orderings, leading to awkwardness in interpreting inequality signs. So, reluctantly, but in conformity with the notation of [3] and [5]-[7], I am using the above definitions.)

It is well known and easy to show (by compactness, in one form or another) that the inverse limit of an inverse system of *finite* nonempty sets is always nonempty. It is also well known that without the hypothesis of finiteness this fails. For instance, suppose S is a set and I some family of subsets of S which forms a directed set under inclusion, and we define an inverse system by letting $X_i = S - i$ (the relative complement of the subset i in S), and for $i \geq j \in I$ letting X_{ji} be the inclusion of X_i in X_j. We see that $\varprojlim_I X_i$ can be identified with the intersection $\bigcap_{i\in I} S - i$, so that if, for instance, S is infinite and we take I to consist of its finite subsets, the X_i are all nonempty but have empty inverse limit. (The above is the one example in this note where it would have been more convenient to use the opposite of the present convention on ordering; we could then have taken I to consist of the *cofinite* subsets of S, and $X_i = i$.)

In fact, every directed partially ordered set I with no greatest element indexes a system of nonempty sets X_i and inclusion maps X_{ji} with empty inverse limit, defined by taking $X_i = \{j \in I \mid j \geq i\}$.

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In the above examples, it is the nonsurjectivity of the X_{ji} that leads to the empty inverse limits, so we might hope for better behavior in systems with surjective maps. Henkin [3] analyzes this situation, and shows that a directed partially ordered set I has the property that every inverse system of nonempty sets and surjective maps indexed by I has nonempty inverse limit if and only if I has cofinality $\aleph_0$ or 1. The hard part of that result is a construction associating to every I of uncountable cofinality an inverse system with surjective maps but empty inverse limit; I will give a simplified version of the construction below.

The sets X_i of that construction (in both [3] and my version) are uncountable. This, and the fact that no such example can occur with finite X_i, leads to the question of whether it can occur with all X_i countable. Examples have been given showing that this, too, can happen. Taking them in reverse chronological order, W. Waterhouse, in the 4-sentence paper [6], notes that we can get an example by taking any uncountable set S, letting I be the set of finite subsets of S ordered by inclusion, letting X_i, for each $i \in I$, be the set of one-to-one maps of i into ω, and letting the X_{ji} be the restriction maps. Higman and Stone [5] give a more complicated example, recalled below, with $I = \omega_1$, the first uncountable ordinal, and show that the properties of this index set allow one to obtain from that example inverse systems of nontrivial *groups, rings,* etc., with surjective maps but trivial inverse limits. I am grateful to P. B. Levy for pointing out that Higman and Stone's result, minus the algebraic applications, was anticipated in the 1930's by the construction of "Aronszajn trees" in [4], namely, trees (there called "tableaux ramifiés", i.e., "branched diagrams") of height ω_1 (there denoted Ω) in which every element has elements above it at every height, yet there are no branches of full height ω_1. (G. Kurepa [4, p.132] says the construction was communicated to him by N. Aronszajn around 1936. It is much lengthier than the later construction of Higman and Stone, and I confess that I have not read it.)

Below, I show that Higman and Stone's method leads to examples of the same phenomena with a much larger class of index sets, properly including all directed partially ordered sets of cofinality $\aleph_1$. I also note examples showing that in the nontriviality result for inverse limits of finite nonempty sets, one cannot replace "finite set" with "transitive G-set" or "finitely generated module"; and I record some questions that our results leave open.

In its present form, this is a discussion paper. If I submit it for publication, I probably should drop some of the commentary, as well as the proof of the Aronszajn-Higman-Stone result (Theorem 2), which does not improve on Higman and Stone's proof, as well as any other results that I learn are not new; and, perhaps, familiar definitions and examples. I welcome readers' thoughts on what to keep.

2. Main results.

In what follows, we will regard a partially ordered set $(I, \leq)$ as a category with morphisms going from larger to smaller elements (the opposite of the standard convention), so that we can call a system of sets and morphisms as in (1)-(5) a functor $X : I \to \mathcal{S}et$.

Recall that for I a partially ordered set, a *chain* in I means a totally ordered subset, and a *down-set* means a subset which, if it contains an element, contains all smaller elements. By a "down-subset" of a subset x of I, we will mean a subset $y \subseteq x$ which is a down-set within x (but not necessarily within I); i.e., such that if $i \in y$, and $j < i$ with $j \in x$, then $j \in y$.

Here, now, is our simplified version of Henkin's construction.

Theorem 1 (Henkin [3])**.** *Let I be a directed partially ordered set of uncountable cofinality. Then there exists a functor $X : I \to \mathcal{S}et$ such that the sets X_i are all nonempty, and the morphisms X_{ji} are surjective, but $\varprojlim X$ is empty.*

Proof. For each $i \in I$, let X_i be the set of all finite chains $x \subseteq I$ with the property that one and only one element of x (necessarily the greatest) is $\geq i$. Given $x \in X_i$ and $j \leq i$, let $X_{ji}(x) \in X_j$ be the down-subset of x gotten by dropping from x all but the smallest of the elements $\geq j$.

It is immediate, without any assumption on the cofinality of I, that the sets X_i are nonempty, and that the X_{ji} satisfy (3)-(5) and are surjective. Now suppose that $x = (x_i)_{i \in I} \in \varprojlim_I X$. Since I is directed, for all $i, j \in I$ there exists a common upper bound k for these indices; it follows that the chains x_i and x_j are down-subsets of the common overchain x_k. Hence one of x_i, x_j must be a down-subset of the other; hence the set of chains $\{x_i \mid i \in I\}$ is totally ordered by inclusion; hence their union is a chain x_∞. In x_∞, every element belongs to a down-subset x_i which is finite, hence distinct elements have distinct (finite) numbers of elements below them, so x_∞ is countable. Moreover, for every $i \in I$, the set x_i, and hence also x_∞, contains an element $\geq i$, so the countable chain x_∞ is cofinal in I. Thus, the existence of an element

$x \in \varprojlim_I X$ implies that I has countable cofinality; so if I has uncountable cofinality, $\varprojlim_I X$ is empty, as claimed. □

(The converse result, that if X is an inverse system with surjective maps, indexed by a directed partially ordered set I of *countable* cofinality, then $\varprojlim_I X$ maps surjectively to each X_i, is standard and easily proved; cf. [3] and [7, Lemma 7].)

T. Kepka (personal communication) calls an inverse system with nonempty objects and surjective maps but empty inverse limit (or, in the case of systems of algebras, the same with "nonempty" and "empty" replaced by "nontrivial" and "trivial", in the sense appropriate to the sort of algebra in question,) a "Henkin spectrum". (He uses "spectrum" for "inverse system".) Naming the phenomenon after Henkin is appropriate if it proves important enough to have a name; but my preference is always to minimize the burden of definitions on the reader, so for now, I will continue to refer to such objects by stating the properties in question.

Now for the Aronszajn-Higman-Stone result.

Theorem 2 (Aronszajn [4]; proof following Higman and Stone [5])**.** *There exists a functor $Y : \omega_1 \to \mathcal{S}et$ such that the sets Y_i are all nonempty* and countable*, and the morphisms Y_{ji} are all surjective, but $\varprojlim Y$ is empty.*

Proof. We start by defining a functor $W : \omega_1 \to \mathcal{S}et$ having weaker properties. For each $i \in \omega_1$, let W_i be the set of all one-to-one maps y of i into the set $\mathbb{Q}$ of rational numbers, such that the image of y is bounded above in $\mathbb{Q}$. For $i \geq j$, let W_{ji} be the restriction map from functions on i to functions on j; these maps W_{ji} are easily seen to be surjective; but a member of their inverse limit would determine a one-to-one map $\omega_1 \to \mathbb{Q}$, which is impossible, so that inverse limit is empty.

The sets W_i are uncountable, but I claim we can find a subfunctor, i.e., a system of subsets $Y_i \subseteq W_i$ respected by the maps W_{ji}, such that each Y_i is countable, and the restrictions Y_{ji} of the maps W_{ji} are still surjective; in fact, such that given any $i > j$ in ω_1, any $y \in Y_j$ whose image is bounded by a rational number r, and any $\epsilon > 0$, we can find $y' \in Y_i$ carried by Y_{ji} to y, such that the image of y' is bounded by $r + \epsilon$.

Indeed, let $i \in \omega_1$ and suppose subsets $Y_j \subseteq W_j$ have been defined for all $j < i$ so that the above conditions hold for these subsets. If i is a successor ordinal $i = j+1$, then we construct Y_i simply by choosing, for each $y \in Y_j$, each rational upper bound r for the image of y, and each rational number $\epsilon > 0$, an element y' extending y, by sending $j \in i$ to any rational number in the interval $(r, r+\epsilon)$; and we let Y_i consist of the countably many elements so chosen. If i is a limit ordinal, then since it is countable, we can write it as the supremum of an ω-indexed chain of smaller ordinals, $j_0 < j_1 < \dots < j_n < \dots$. Now for every $n \in \omega$, every $y \in Y_{j_n}$, every rational upper bound r for the image of y, and every rational number $\epsilon > 0$, let us write $y_n = y$, then lift this to an element $y_{n+1} \in Y_{j_{n+1}}$ whose image is bounded above by $r + \epsilon/2$, as we may by our inductive hypothesis; then lift that to an element $y_{n+2} \in Y_{j_{n+2}}$ whose image is bounded above by $r + 3\,\epsilon/4$, and so forth. Since each y_{n+m} is an extension of the one before, their union will be a one-to-one map $y' : i \to \mathbb{Q}$ whose image is bounded above by $r + \epsilon$. Making one such construction for each choice of y, n, r and ϵ, we get countably many such elements y', and we again define Y_i to be the set consisting of these.

The resulting subfunctor Y inherits from W the property of having empty inverse limit, and by construction, the sets Y_i are countable and the maps between them surjective, as claimed. □

(In [5] Higman and Stone used, in place of our W_i, the set of all *isotone* bounded embeddings of i into the *real* numbers, getting emptiness of the inverse limit from the fact that the real line has countable cofinality.)

We shall now extend this result to systems indexed by certain not necessarily totally ordered sets, with the help of part (i) of the next lemma. Part (ii) is recorded for its interest as a partial converse to (i). The lemma seems likely to be known; I hope that anyone who recognizes it will let me know.

Lemma 3. *Let κ be an infinite cardinal, and I a directed partially ordered set of cofinality κ. Then*

(i) *There exists a surjective isotone map $I \to \kappa$.*

(ii) *There does not exist any surjective isotone map from I to a regular cardinal $> \kappa$.*

Proof. To show (i), it suffices to find an isotone map $f : I \to \kappa$ whose image has cardinality κ, since that image will be order-isomorphic to κ.

By assumption, there exists a set map $g : \kappa \to I$ with cofinal image. For each $i \in I$, let $f(i)$ be the least $\alpha \in \kappa$ such that $g(\alpha) \geq i$. The map $f : I \to \kappa$ is clearly isotone. Moreover, every $i \in I$ is majorized by $g(f(i))$, hence the image set $gf(I)$ is cofinal in I, hence it has cardinality $\geq \kappa$. Hence $f(I) \subseteq \kappa$ has cardinality $\geq \kappa$, hence equal to κ, as required.

To get (ii), suppose f is an isotone map from I into a regular cardinal $\lambda > \kappa$, and $J \subseteq I$ is a cofinal subset of I of cardinality κ. Then $f(J)$, being a subset of the regular cardinal λ having cardinality $< \lambda$, is bounded above by some $\alpha \in \lambda$. Since J is cofinal in I, the image under f of any element of I is also bounded by α; so f is not surjective. $\square$

The above lemma will be used in combination with the next result, which is well-known in the case where I is a subset of J with the induced ordering, and $f : I \to J$ is the inclusion; I would be interested to know whether it has been noted in more general contexts. We state this lemma in its natural category-theoretic generality, though we will only use the case $\mathcal{C} = \mathcal{S}et$, and, as sketched in the first paragraph of the proof, the general case actually reduces to that one.

Lemma 4. *Let $f : I \to J$ be an isotone map of directed partially ordered sets such that $f(I)$ is cofinal in J, and let $Y : J \to \mathcal{C}$ be a functor. Then $\varprojlim_I Y \circ f = \varprojlim_J Y$, in the sense that if either limit exists, so does the other, and the natural morphism $\varprojlim_J Y \to \varprojlim_I Y \circ f$ is an isomorphism.*

Proof. By definition, $\varprojlim_I Y \circ f$ and $\varprojlim_J Y$ denote representing objects for certain contravariant "set-of-all-cones" functors on $\mathcal{C}$, so what must be proved is that those two functors are isomorphic. The descriptions of the values of those functors at an object of $\mathcal{C}$ are equivalent to the concrete description (6) of the limits of certain $\mathcal{S}et$-valued functors on I, respectively J. Examining these, one finds that the general case of the lemma reduces to the case $\mathcal{C} = \mathcal{S}et$; so we shall assume Y a $\mathcal{S}et$-valued functor.

Now every element $x = (x_j)_{j\in J} \in \varprojlim_J Y$ induces an element $x \circ f = (x_{f(i)})_{i\in I} \in \varprojlim_I Y \circ f$; what we have to show is that this map $-\circ f : \varprojlim_J Y \to \varprojlim_I Y \circ f$ is bijective. Given $x = (x_i)_{i\in I} \in \varprojlim_I Y \circ f$, let us define $x' \in \varprojlim_J Y$ as follows. For each $j \in J$, we can, by hypothesis, find $i \in I$ such that $f(i) \geq j$; let $x'_j = Y_{j,f(i)}(x_i)$. To show this well-defined, suppose $f(i)$ and $f(i')$ both majorize j. By directedness of I, we can find $i'' \in I$ majorizing both i and i'. Since $x \in \varprojlim_I Y \circ f$, we have $x_i = Y_{f(i),f(i'')}(x_{i''})$ and $x_{i'} = Y_{f(i'),f(i'')}(x_{i''})$, so the two candidates for x'_j namely $Y_{j,f(i)}(x_i)$ and $Y_{j,f(i')}(x_{i'})$, both reduce to $Y_{j,f(i'')}(x_{i''})$, and so are equal. It is straightforward to verify that the element $x' = (x'_j)_{j\in J}$ indeed lies in $\varprojlim_J Y$, and that this construction gives, as required, a two-sided inverse to $-\circ f : \varprojlim_J Y \to \varprojlim_I Y \circ f$. $\square$

We can now get our main result.

Theorem 5. *Let I be any directed partially ordered set which admits an isotone map onto ω_1 (for instance, by Lemma 3, any partially ordered set of cofinality ω_1). Then there exists a functor $X : I \to \mathcal{S}et$ such that the sets X_i are countable and nonempty, and the morphisms X_{ji} are surjective, but $\varprojlim X = \emptyset$.*

Proof. Let $X = Y \circ f$, where $f : I \to \omega_1$ is a surjective isotone map, and $Y : \omega_1 \to \mathcal{S}et$ is the functor of Theorem 2. By Lemma 4, $\varprojlim_I X = \varprojlim_{\omega_1} Y = \emptyset$. $\square$

3. Remarks and questions.

The construction of Theorem 2 begins, as we saw, with a functor W whose values, like those of the functor X of Theorem 1, are uncountable sets. What property of W allowed us to "cut it down" to a countable-set-valued functor, while preserving surjectivity of the connecting maps? If for each $r \in \mathbb{Q}$ and $i \in \omega_1$ we let $W_i(r)$ denote the subset of W_i consisting of those maps whose images are bounded above by r, we can express the property we used as saying that W is the union of a $\mathbb{Q}$-indexed chain of subfunctors $W(r)$, such that

(7) Whenever $i \in \omega_1$ is the supremum of an ω-indexed chain $j_0 < j_1 < \cdots < j_n < \cdots$, and $s \in \mathbb{Q}$ is the supremum of an ω-indexed sequence $r_0 < r_1 < \cdots < r_n < \cdots$, then any element of $\varprojlim_{n\in\omega} W_{j_n}(s)$ whose image in each W_{j_n} lies in $W_{j_n}(r_n)$ arises from an element of $W_i(s)$.

If we had started with a functor on ω_1 with the simpler property

(8) Whenever $i \in \omega_1$ is the supremum of an increasing chain $j_0 < j_1 < \cdots < j_n < \cdots$, the natural map $W_i \to \varprojlim_{n\in\omega} W_{j_n}$ is surjective,

this would have been too strong: It would have implied that the maps $\varprojlim_{\omega_1} W \to W_i$ $(i \in \omega_1)$ were all surjective, by essentially the same "successive lifting" argument that shows that for any set-valued functor X with surjective connecting maps on a directed partially ordered set of countable cofinality, the maps $\varprojlim X \to X_i$ are surjective, using (8) to carry us past the limit ordinals. So $\varprojlim_{\omega_1} W$ would not have been empty, which was one of the key ingredients of the construction. The weaker statement (7) allowed us to lift elements past any countable number of limit steps, without forcing the same sort of lifting to work through all $\aleph_1$ steps.

In the system X of Theorem 1, on the other hand, there is no evident way to find an analog of the family of subfunctors $W(r)$. So we ask

Question 6. *Does the functor X given by the $I = \omega_1$ case of the proof of Theorem 1 have a subfunctor Y with all Y_i countable and nonempty, and all Y_{ji} surjective? Does it in fact have a $\mathbb{Q}$-indexed chain of subfunctors satisfying* (7)*?*

It would also be interesting to know whether in Theorem 2, and hence in Theorem 5, one can replace ω_1 with other uncountable regular cardinals. I twice believed I could prove that theorem with ω_1 replaced by any infinite successor cardinal κ^+, and "countable set" replaced by "set of cardinality $\leq \kappa$". The idea was to replace the set $\mathbb{Q}$ by a totally ordered set Q_κ having cardinality κ and no greatest element, such that every open interval in Q_κ had a subset order-isomorphic to κ. (This is not hard to construct.) If for each $i \in \kappa^+$ one then lets W_i denote the set of maps $i \to Q_\kappa$ whose image is bounded above in Q_κ and if one defines the subsets $W_i(r)$ $(r \in Q_\kappa)$ as above, one does get the analog of (7) for sequences indexed by cardinals $\leq \kappa$. However, if one attempts to construct subsets Y_i by a recursive process analogous to that used in the proof of Theorem 2, a problem arises whenever i has uncountable cofinality. The analog of the ω-indexed chain $j_0 < j_1 < \cdots < j_n < \cdots$ with supremum i will be a chain indexed by $\mathrm{card}(i)$, and the process of lifting our element through this chain is not as easy as before: there are uncountably many limit steps along the way. Though at each such step we can lift a sequence of previous choices to get an element of $W_j(r_j)$, I see no way of assuring that this limit could be taken to lie in $Y_j(r_j)$. So we ask

Question 7. *Does the analog of Theorem 2 hold if one replaces countable sets, and ω_1, by sets of cardinality $\leq \kappa$, and κ^+, for an arbitrary infinite cardinal κ, or at least for* some *such cardinals other than ω? Can we perhaps even replace ω_1 by other regular cardinals* without *weakening the condition "countable"?*

The directed sets to which Theorem 5 is applicable as it stands, namely those admitting isotone maps onto ω_1, already form a much larger class than those of cofinality ω_1. An easy class of examples with cofinalities $> \omega_1$ is gotten by taking the direct product of ω_1 with any directed set of larger cofinality, under the product ordering (componentwise inequality). For a less obvious case, let κ be any cardinal $> \omega_1$, and I the partially ordered set of all countable subsets of κ, ordered by inclusion. The cofinality of I is at least κ (since if we merely seek to majorize the singleton elements of I, no element of I majorizes more than countably many of them, hence at least κ elements are needed). However, for any $i \in I$, the order-type of i as a subset of κ is the order-type of a unique element $f(i) \in \omega_1$; and this function f is easily seen to be an isotone surjection $I \to \omega_1$.

4. Inverse limits of algebras.

Let $\mathbf{V}$ be a variety of algebras in the sense of universal algebra, with all operations finitary, and let

(9) $\quad F: \mathcal{S}et \to \mathbf{V}$

be the free $\mathbf{V}$-algebra functor. Assume that

(10) if $\mathbf{V}$ has any constant derived operations, it has zeroary operations.

In this situation, we can associate to each element a of a free algebra $F(S)$ its "support",

(11) $\quad \mathrm{supp}(a) =$ the least subset $A \subseteq S$ such that $a \in F(A) \subseteq F(S)$,

a finite subset of S.

This operator has the rather eccentric property that given a set map $u : S \to T$ and an element $a \in F(S)$, the set $u(\mathrm{supp}(a))$ is not a subset of $\mathrm{supp}(F(u)(a))$, but a superset thereof,

(12) $\quad u(\mathrm{supp}(a)) \supseteq \mathrm{supp}(F(u)(a))$.

This inclusion can be proper, since elements of $u(\mathrm{supp}(a))$ may "vanish" from $\mathrm{supp}(F(u)(a))$ as a result of cancellations that occur when elements of S fall together under u. Hence, given an inverse system X in $\mathcal{S}et$ and an element $a = (a_i)_{i \in I} \in \varprojlim_I F(X_i)$, it will not in general be true that the finite sets $\mathrm{supp}(a_i) \subseteq F(X_i)$ $(i \in \omega_1)$ form a subfunctor of X. If it were, then in the case where $\varprojlim_I X_i = \emptyset$, we could deduce that $\varprojlim_I F(X_i)$ was the algebra $F(\emptyset)$, thus getting "trivial" algebras as inverse limits of free algebras subject to the same sorts of conditions under which we got $\emptyset$ as an inverse limit of sets.

Nevertheless, Higman and Stone [5] note that if X is an inverse system of sets *indexed by* ω_1 and having empty inverse limit, and we apply F to it, we do get an inverse system of algebras with inverse limit $F(\emptyset)$. To see this, consider, more generally, any inverse system X of sets whose index set I has the property that every countable subset of I has an upper bound in I, and suppose $a \in \varprojlim F(X_i)$. If the set of integers $\mathrm{card}(\mathrm{supp}(a_i))$ $(i \in I)$ were unbounded, then letting j be an upper bound for some countable set on which they assumed arbitrarily large values, we would get a contradiction to the statement that $\mathrm{supp}(a_j)$ is finite. Hence those cardinalities must, rather, assume a maximum value at some $i_0 \in I$. We then see from (12) that on the set $\{i \mid i \geq i_0\} \subseteq I$, the maps X_{ij} $(i_0 \leq i \leq j)$ carry $\mathrm{supp}(a_j)$ to $\mathrm{supp}(a_i)$ bijectively, and it follows that every element of $\mathrm{supp}(a_{i_0})$ induces an element of $\varprojlim_I X_i$. We deduce that if that limit is empty, $\varprojlim_I F(X_i)$ is indeed $F(\emptyset)$.

Combining this construction with Lemma 4, we get

Corollary 8. *If I is a directed partially ordered set admitting an isotone map onto ω_1, and $\mathbf{V}$ a variety of finitary algebras satisfying* (10)*, then there exists a functor $X : I \to \mathbf{V}$ such that the X_i are free algebras on nonempty countable sets, and the homomorphisms X_{ji} are surjective, but $\varprojlim X$ is the free algebra on the empty set.* □

This leaves unanswered

Question 9. *Suppose I is a directed partially ordered set of uncountable cofinality* (*but not assumed to have upper bounds on countable subsets*)*, and $\mathbf{V}$ a variety of finitary algebras satisfying* (10)*.*

(i) *If $X : I \to \mathcal{S}et$ is a functor having empty inverse limit, and $F : \mathcal{S}et \to \mathbf{V}$ is the free $\mathbf{V}$-algebra functor, must $\varprojlim_I F \circ X \cong F(\emptyset)$?*

If the answer to (i) *is negative, we may still ask,*

(ii) *Does there exist some functor $X : I \to \mathbf{V}$ such that the connecting morphisms X_{ji} are all surjective, but the natural maps from $\varprojlim X$ to the algebras X_i are not all surjective?*

One approach I thought might be applicable to (i) above is the following. Assume we have an element $a \in \varprojlim F(X_i)$ whose image in some $F(X_{i_0})$ does not lie in the subalgebra $F(\emptyset)$. Then let us associate to every $i \geq i_0$ the finite nonempty set given by the support of the image of a in $F(X_i)$, and try to find a subsystem of this system of finite sets consisting of singletons, thus obtaining an element of $\varprojlim X$.

Unfortunately, there may not be such a subsystem – even if I is finite! For instance, let $\mathbf{V}$ be the category of groups, let I be the 4-element lattice 2×2, let the set associated with the top element be $\{x, y, z\}$, and in the free group on that set, let $a = xy^{-1}z$. Let the sets associated with the two elements at the middle level of I be obtained from the top set by the identifications $x = y$ and $y = z$ respectively, and let the bottom set be obtained by identifying all three generators. Then we see that the support of a is $\{x, y, z\}$, while we find that the supports of its images in the other three groups are the singletons $\{z\}$, $\{x\}$, and $\{x = y = z\}$; so there is no element of $\mathrm{supp}(a)$ that is mapped to members of all these sets.

(In this example, the "disappearance" of generators from our support-sets was based on those generators falling together with other generators. But an element can also disappear because two *different* elements fall together. E.g., taking $xyz^{-1}x^{-1}y$ in the free group on $\{x, y, z\}$, if y and z fall together, then x is lost from the support.)

Here is an example of an inverse system of sets with surjective maps and empty inverse limit, together with a family of finite subsets of the members of this inverse system, which we might try to realize in this way as supports of elements of free algebras. Let us take any uncountable set S, let I consist of the finite subsets of S, and, as in [6], let each X_i $(i \in I)$ be the set of one-to-one maps from i to ω. Within each X_i, let Y_i be the finite set consisting of those maps whose image is an initial segment of ω (necessarily, the natural number $\mathrm{card}(i)$). Then the maps $X_j \to X_i$ indeed take each Y_j to a superset of Y_i. This example itself can't be realized as desired, because when j contains just one more element than i, the map $X_j \to X_i$ happens to be one-to-one on elements having images in Y_j; so we have elements "vanishing" without any elements "falling together", which doesn't happen to supports of elements of free algebras. However, by slight modifications of this example – e.g., restricting I to sets i of even cardinality; or keeping I as above, but weakening the restriction defining the elements of Y_i to say that their images lie in the initial segment of ω of cardinality $2\,\mathrm{card}(i)$ – one gets cases for which it seems plausible that the system could be so realized.

5. G-sets and R-modules.

Tomáš Kepka (personal communication) has asked whether an inverse limit of nonzero *finitely generated* modules and surjective homomorphisms can be the zero module. We shall obtain examples of this by first constructing analogous examples for group actions.

Recall that a G-set is *transitive* if and only it is generated by a single element.

Lemma 10. *There exists a group G and an inversely directed system of transitive G-sets and surjective G-set homomorphisms, indexed by ω_1, having empty inverse limit. Moreover* (*with considerable additional work*), *G can be taken abelian.*

Proof. Example 1, nonabelian, easy to describe. We start with a variant of the set-theoretic construction of [6]: Let $X : \omega_1 \to \mathcal{S}et$ carry every $i \in \omega_1$ to the set of all one-to-one functions $i \to \omega$ whose images are not cofinite, and let the X_{ji} $(i \geq j)$ be the restriction maps. Clearly, this set-valued functor again has surjective maps X_{ji}, but empty inverse limit.

Now let G be the symmetric group on ω (the group of *all* permutations of that set). Then G acts naturally on the left on each X_i, making it a transitive G-set, and these G-set structures are respected by the maps X_{ji}; so X can be regarded as an inverse system of cyclic G-sets with empty inverse limit.

Example 2, with G abelian. Here we will obtain our G as a subgroup of a larger abelian group, namely

$$H \;=\; \mathbb{Z}^{\omega_1 \times \omega_1}. \tag{13}$$

Let us define elements $c_{ij} \in H$ for all $i, j \in \omega_1$ by

$$c_{ij}(i', j') \;=\; \begin{cases} 1 & \text{if } i' < i \text{ and } j' < j, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

(The set at which c_{ij} has the value 1 can be pictured as a "rectangle"; but note that the height and width of this rectangle, though countable, are not, in general, finite.)

We now define

$$G \;=\; \text{the subgroup of } H \text{ generated by } \{c_{ji} - c_{ii} \mid i < j \in \omega_1\}. \tag{15}$$

Thus, the support of each generator $c_{ji} - c_{ii}$ is a rectangle with upper left corner at $(i, i-1)$ (just below the diagonal), upper right corner at $(j-1, i-1)$, and lower corners at $(i, 0)$ and $(j-1, 0)$.

By a "cross-section" of an element $h \in H$, we shall mean the element of $\mathbb{Z}^{\omega_1}$ gotten, in the obvious way, from the coordinates of h by fixing any $j \in \omega_1$, and letting i vary. We see that

(16) every element of G has only finitely many distinct cross-sections,

(17) every cross-section of an element of G has bounded support; i.e., its i-th coordinate is 0 for all sufficiently large $i \in \omega_1$,

(18) all elements of G have value zero on and above the diagonal of $\omega_1 \times \omega_1$.

(The construction below would work if we instead took G to consist of all elements of H satisfying the conclusions of (16)-(18); or any group between the group defined in (15) and that one.)

For each $i \in \omega_1$, let us now define

$$H_i \;=\; \mathbb{Z}^{\omega_1 \times i}. \tag{19}$$

Thus we have obvious coordinate-dropping homomorphisms,

$$f_i : H \to H_i, \quad f_{ij} : H_j \to H_i \ \ (i \leq j). \tag{20}$$

The homomorphisms f_i allow us to regard each H_i as an H-set, and hence as a G-set. We now define, within each H_i, the G-orbit (i.e., coset of $f_i(G)$),

$$X_i \;=\; f_i(c_{ii} + G) \quad \text{(see (14))}. \tag{21}$$

Clearly all elements of X_i satisfy the conclusions of (16) and (17) (with "cross-section" defined for elements of H_i as for elements of H); while rather than (18), we see that they satisfy

(22) all elements of $f_i(c_{ii} + G)$ have value 1 on and above the diagonal.

I claim now that for $i \leq j$, the map $f_{ij} : H_j \to H_i$ carries the orbit X_j to the orbit X_i. Indeed, $f_{ij}(f_j(c_{jj})) = f_i(c_{jj})$, and since f_i discards coordinates with second subscript $\geq i$, this equals $f_i(c_{ji})$. Since $c_{ji} - c_{ii} \in G$, the G-orbit of this element contains $f_i(c_{ii})$, and so equals X_i.

Note that $\varprojlim_{\omega_1} H_i = H$, hence that $\varprojlim_{\omega_1} X_i$ can be identified with a subset of H. Suppose that subset were nonempty; say $y \in \varprojlim_{\omega_1} X_i$. Then as a function on $\omega_1 \times \omega_1$, y inherits from the X_i the conclusions of (17) and (22), while the property (16) of the X_i implies that below each horizontal line in $\omega_1 \times \omega_1$, y has only finitely many distinct cross-sections. But since the ordinal ω_1 has uncountable cofinality, this implies that y has only finitely many cross-sections altogether; i.e., does in fact satisfy (16).

Since each of these finitely many cross-sections has bounded support, there is some $i_0 \in \omega_1$ containing all their supports. Applying this fact to the i_0-th cross-section, we see that the (i_0, i_0) coordinate of y must be 0, contradicting (22), and completing the proof that the system of G-sets X has empty inverse limit. $\square$

We mention some easy variants of the above examples. In the noncommutative example we could (in the spirit of Higman and Stone's original proof of Theorem 2) let $X : \omega_1 \to \mathcal{S}et$ take each $i \in \omega_1$ to the set of *order*-embeddings of i in $\mathbb{Q}$ (or $\mathbb{R}$) whose image is bounded above, and let G be the group of all order-automorphisms of $\mathbb{Q}$ (respectively $\mathbb{R}$). (Or, if we wish, the subgroup thereof consisting of those order-automorphisms that act as the identity on all sufficiently large arguments.) In the commutative example, we could, of course, replace the $\mathbb{Z}$ in $\mathbb{Z}^{\omega_1 \times \omega_1}$ by any nonzero abelian group A, and the 1 in (14) by any nonzero element of A.

To get the module-theoretic result Kepka asked for, let G and X be as in the statement of the above lemma, let k be any field, and let F be the free k-module functor, which associates to every set a vector space with that set as basis. Then $F \circ X$ gives a functor from ω_1 to k-vector-spaces, with surjective connecting maps, which (by the result of Higman and Stone by which we led up to Corollary 8) has the zero vector space as inverse limit.

Combining the action of G on X with the vector-space-valued functor F, we see that $F \circ X$ can be made a functor from ω_1 to left modules over the group algebra $k\,G$. For each i, any element of X_i generates $F(X_i)$ as a $k\,G$-module, so these modules are cyclic. Taking $R = k\,G$, we get

Corollary 11. *There exists a ring R, and an inversely directed system of cyclic (in particular, finitely generated) nonzero left R-modules, and surjective R-module homomorphisms, whose inverse limit is the zero R-module. In fact, R can be taken commutative.* $\square$

From examples of this phenomenon for one ring, group, etc., one can get such examples for others: If a ring R admits an inverse system X as in Corollary 11, and S is an over-ring which is free as a *right* R-module, then applying extension of scalars to X, one finds that S also admits such a system. The analogous statement for a monoid and overmonoid is likewise clear; applying this fact about monoids to a group G, we see that if G has the property of Lemma 10 then *every* overgroup inherits that property. The property is also clearly inherited by rings, groups or monoids that admit homomorphisms *onto* rings, groups or monoids with the property; so in particular, all free rings, free groups and free monoids of large enough rank have the property. Finally, if a ring, group or monoid R, G or M has a subring, subgroup, or submonoid R', G' or M' such that R, G or M is finitely generated as a left R'-module, G'-set or M'-set, then every finitely generated left R-module, G-set or M-set remains finitely generated over R', G' or M', so the property carries over from R, G or M to R', G' or M'.

Question 12. *Can a* countable *ring R admit an inverse system of finitely generated nonzero left R-modules and surjective module homomorphisms with zero inverse limit?*

Can a countable monoid M admit an inverse system of finitely generated nonempty left M-sets and surjective M-set homomorphisms with empty inverse limit?

Can a countable group G admit an inverse system of transitive left G-sets and G-set homomorphisms with empty inverse limit?

Other examples of trivial module-theoretic inverse limits are obtained in [2, §7-§8].

In [1, Theorem 4], we use the result of Aronszajn-Higman-Stone (Theorem 2 above) in a slightly more subtle fashion than here, to get inverse systems of modules and surjective homomorphisms with a different contrast between the properties of the given modules and of their inverse limit.

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