Comments, corrections, and related references welcomed, as always!

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COMMUTING MATRICES, AND MODULES OVER ARTINIAN LOCAL RINGS

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Abstract. Gerstenhaber [5] proves that any commuting pair of $n \times n$ matrices over a field k generates a k-algebra A of k-dimension $\leq n$. A well-known example shows that the corresponding statement for 4 matrices is false. The question for 3 matrices is open.

Gerstenhaber's result can be looked at as a statement about the relationship between the length of a 2-generator finite-dimensional commutative k-algebra A, and the lengths of faithful A-modules. Wadsworth [19] generalizes this result to a larger class of commutative rings than those generated by two elements over a field. We recover his result, with a slightly improved argument.

We then explore some examples, raise further questions, and make a bit of progress toward answering some of these.

When I drafted this note, I thought the main result, Theorem 3, was new; but having learned that it is not, I probably won't submit this for publication unless I find further strong results to add. However, others may find interesting the observations, partial results, and questions noted below, and perhaps make progress on them.

(An earlier version of this note had an appendix on generation and subdirect decompositions of modules over not necessarily commutative Artinian rings. This has been moved to [3].)

1. Wadsworth's generalization of Gerstenhaber's result

To introduce Wadsworth's strengthening of the result of Gerstenhaber quoted in the first sentence of the abstract, note that a k-algebra of $n \times n$ matrices generated by two commuting matrices can be viewed as an action of the polynomial ring $k[s, t]$ on the vector space k^n. A class of rings generalizing those of the form $k[s, t]$ are the rings $R[t]$ for R a principal ideal domain. The analog of an action of $k[s, t]$ on a finite-dimensional k-vector space is then an $R[t]$-module of finite length. A key tool in our study of such actions will be

Corollary 1 (to the Cayley-Hamilton Theorem; cf. [19, Lemma 1])**.** *Let R be a commutative ring, M an R-module which can be generated by $n < \infty$ elements, and f an endomorphism of M. Then f^n is an R-linear combination of $1_M,\ f,\ \ldots,\ f^{n-1}$.*

Proof. Write M as a homomorphic image of R^n. Since R^n is projective, we can lift f to an endomorphism g thereof. Since g satisfies its characteristic polynomial, f satisfies the same polynomial. $\square$

This shows that the unital subalgebra of $\mathrm{End}_R(M)$ generated by f will be spanned over R by the f^i with $0 \leq i < n$, but doesn't say anything about the size of the contribution of each f^i. In the next lemma, we obtain information of that sort, by re-applying the above corollary to various i-generator submodules of M.

Lemma 2. *Let R be a commutative ring, M a finitely generated R-module, and f an endomorphism of M. For each $i \geq 0$, let I_i be the ideal of R generated by all elements u such that $u\,M$ is contained in an f-invariant submodule of M which is i-generated as an R-module. (So $I_0 \subseteq R$ is the annihilator of M, and we have $I_0 \subseteq I_1 \subseteq \ldots$, with the ideals becoming R once we reach an i such that M is i-generated.)*

Let A be the unital R-algebra of endomorphisms of M generated by f. For each $i \geq -1$, let A_i be the R-submodule of A spanned by $\{1,\ f,\ f^2,\ \ldots,\ f^i\}$. Then

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(i) *For all* $i \geq 0$, *the* R*-module* A_i/A_{i-1} *is a homomorphic image of* R/I_i. *Hence*

(ii) *If* M *has finite length, and can be generated by* d *elements, then* A *has length* $\leq \sum_{i=0}^{d-1} \mathrm{length}(R/I_i)$ *as an* R*-module.*

Proof. Since $A_i = Rf^i + A_{i-1}$, (i) will follow if we show that $I_i f^i \subseteq A_{i-1}$, which by definition of I_i is equivalent to showing that, for each $u \in R$ such that uM is contained in an i-generated f-invariant submodule M' of M, we have $uf^i \in A_{i-1}$. Given such u and M', let f' be the restriction of f to M'. By Corollary 1, f'^i is an R-linear combination of the lower powers of f'; say $\sum_{j<i} a_j f'^j$. Hence the restriction of f to $uM \subseteq M'$ satisfies the same relation. This says that $f^i u = \sum_{j<i} a_j f^j u$, equivalently, that $uf^i = \sum_{j<i} ua_j f^j$, showing that $uf^i \in A_{i-1}$, as required.

We deduce (ii) by summing over the steps of the chain $\{0\} = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_{d-1} = A$. □

We can now prove

Theorem 3 (Wadsworth [19, Theorem 1], after Gerstenhaber [5, Theorem 2, p. 245])**.** *Let* M *be a module of finite length over a commutative principal ideal domain* R, *let* f *be an endomorphism of* M, *and let* A *be the unital* R*-algebra of* R*-module endomorphisms of* M *generated by* f. *Then*

(1) $\qquad \mathrm{length}_R(A) \ \leq \ \mathrm{length}_R(M).$

Proof. Because R is a commutative principal ideal domain, we can write M as $R/(q_0) \oplus \cdots \oplus R/(q_{d-1})$, where $q_{d-1} \,|\, q_{d-2} \,|\, \ldots \,|\, q_0$ ([15, Theorem III.7.7, p. 151], with terms relabeled). Note that for $i = 0, \ldots, d-1$, the element q_i annihilates the summands $R/(q_i), \ldots, R/(q_{d-1})$ of the above decomposition. Hence $q_i M$ is generated by i elements, so in the notation of the preceding lemma, $q_i \in I_i$. (In fact, it is not hard to check that $I_i = (q_i)$.) In particular, $\mathrm{length}_R(R/I_i) \leq \mathrm{length}_R(R/(q_i))$.

By that lemma, the length of A as an R-module is $\leq \sum \mathrm{length}_R(R/I_i)$, which by the above inequality is $\leq \sum \mathrm{length}_R(R/(q_i)) = \mathrm{length}_R(M)$. □

We remark that in the above results, the *length* of a module is really a proxy for its image in the Grothendieck group of the category of finite-length modules. I did not so formulate the results for simplicity of wording, and to avoid excluding readers not familiar with that viewpoint. This has the drawback that when we want to pass from that lemma back to Gerstenhaber's result on k-dimensions of matrix algebras, the lengths do not directly determine these dimensions, which the elements of the Grothendieck group would. (E.g., the $\mathbb{R}$-algebras $\mathbb{R}[s]/(s)$ and $\mathbb{R}[s]/(s^2+1)$ both have length 1 as modules, but their $\mathbb{R}$-dimensions are 1 and 2 respectively.)

However, we can get around this by applying Theorem 3 after extension of scalars to the algebraic closure of k. Indeed, if A is a k-algebra of endomorphisms of a finite-dimensional k-vector-space M, then the k-dimensions of A and M are unaffected by extending scalars to the algebraic closure $\bar{k}$ of k; and for k algebraically closed, the length of M as an A-module is just its k-dimension. Hence an application of Theorem 3 with $R = \bar{k}[s]$ to these extended modules and algebras gives us Gerstenhaber's result for the original modules and algebras.

2. Some notes on the literature

The hard part of Gerstenhaber's proof of his result was a demonstration that the variety (in the sense of algebraic geometry) of all pairs of commuting $n \times n$ matrices is irreducible. Guralnick [7] notes that this fact had been proved earlier by Motzkin and Taussky [16], and gives a shorter proof of his own.

The first proof of Gerstenhaber's result by non-algebraic-geometric methods is due to Barría and Halmos [1]. Wadsworth [19] abstracts that proof by replacing $k[s]$ by a general principal ideal domain R. His argument differs from ours in that he obtains Corollary 1 only for R a principal ideal domain and M a torsion R-module. This restriction arises from his calling on the fact that every such module M *embeds* in a free module over some factor-ring of R (in the notation of our proof of Theorem 3, the free module of rank d over $R/(q_0)$), where we use, instead, the fact that any finitely generated R-module is a *homomorphic image* of a free R-module of finite rank, which is true for any commutative ring R. Of course, the restriction assumed in Wadsworth's proof holds in the case to which we both apply the result; but the more general statement of Corollary 1 seemed worth recording.

I mentioned above that where I speak of the length of a module, a more informative statement would refer to its image in a Grothendieck group. Wadsworth uses an invariant of finite-length modules over PIDs that is equivalent to that more precise information: in the notation of our proof of Theorem 3, the equivalence class, under associates, of the product $q_0 \dots q_{d-1} \in R$. He also shows [19, Theorem 2] that as an R-module, A can in fact be embedded in M. (However, we will note at the end of §4 that it cannot in general be so embedded as an $R[t]$-module.)

Gerstenhaber [5] proves a bit more about the algebra generated by two commuting $n \times n$ matrices than we have quoted: he also shows that it is *contained* in a commutative matrix algebra of dimension *exactly* n, at least after a possible extension of the base field. (He mentions that he does not know whether this is true without extension of the base field.) We shall not discuss that property further here.

Guralnick and others [8], [18] have continued the algebraic geometric investigation of these questions. Some investigations which, like this note, focus more on methods of linear algebra are [11], [10], [17]. For an extensive study of the subject, see O'Meara, Clark and Vinsonhaler [14, Chapter 5].

Returning to Theorem 3, the hypothesis that R be a principal ideal domain can be weakened, with a little more work, to say that R is a Dedekind domain, or even a Prüfer domain, since under these assumptions, every finite-length homomorphic image of R is a direct product of uniserial rings, which is what is really needed to get the indicated description of finite-length modules (though I don't know a reference stating this description in those cases). We shall discuss in §§5-6 wider generalizations that one may hope for, and will make some progress in those directions.

But for the next two sections, let us return to commuting matrices over a field, and examine what can happen in algebras generated by more than two such matrices.

3. Counterexamples with 4 generators

The standard example showing that a 4-generator algebra of $n \times n$ matrices can have dimension $> n$ takes $n = 4$, and for A, the algebra of 4×4 matrices generated by e_{13}, e_{14}, e_{23} and e_{24}. These commute, since their pairwise products are zero, and A has for a basis these four elements and the identity matrix 1, and so has dimension $5 > 4 = n$.

If the reader finds it disappointing that the extra dimension comes from the convention that algebras are unital, note that without that convention, one can obtain the same subalgebra from the four generators $1 + e_{13}$, e_{14}, e_{23}, e_{24}, using the fact that the not-necessarily-unital algebra generated by an upper triangular matrix (here $1 + e_{13}$) always contains both the diagonal part (here 1) and the above-diagonal part (here e_{13}) of that matrix.

One can modify this example to get commutative 4-generator matrix algebras in which the dimension of the algebra exceeds the size of the matrices by an arbitrarily large amount. Namely, for any m, let us form within the algebra of $4m \times 4m$ matrices a "union of m copies" of each of the matrix units used in the above example. To do this, let $E_{13} = \sum_{j=0}^{m-1} e_{4j+1,\,4j+3}$, $E_{14} = \sum_{j=0}^{m-1} e_{4j+1,\,4j+4}$, $E_{23} = \sum_{j=0}^{m-1} e_{4j+2,\,4j+3}$, $E_{24} = \sum_{j=0}^{m-1} e_{4j+2,\,4j+4}$, and let us also choose a diagonal matrix D having one value, $\alpha_0 \in k - \{0\}$, in the first four diagonal positions, a different value, $\alpha_1 \in k - \{0\}$, in the next four, and so on; in other words, $D = \sum_{j=0}^{m-1} \alpha_j (\sum_{i=1}^{4} e_{4j+i,\,4j+i})$. (Here we assume $|k| \geq m+1$, so that the α_i can be taken distinct.) We then take as our four generators $D + E_{13}$, E_{14}, E_{23} and E_{24}. From the fact about upper triangular matrices called on in the preceding paragraph, the algebra these generate contains D, E_{13}, E_{14}, E_{23} and E_{24}. Using just D and the k-algebra structure, one gets the m diagonal idempotent elements $\sum_{i=1}^{4} e_{4j+i,\,4j+i}$ $(j = 0, \dots, m-1)$, and with the help of these, one sees that our 4-generator algebra is the direct product of m copies of the algebra of the preceding paragraphs. Thus, we have a 4-generated commutative algebra of dimension $5m$ within the ring of $4m \times 4m$ matrices.

Because of the strong way this construction used a diagonal matrix D, I briefly hoped that if we restricted attention to algebras A generated by four commuting *nilpotent* matrices, the dimension of A might never exceed the size of the matrices by more than 1. But the following family of examples contradicts that guess.

In describing it, I will again use the language of vector spaces and their endomorphisms. Let m be any positive integer, and let M be a $(5m^2 + 3m)/2$-dimensional k-vector-space, with basis consisting of elements which we name (proactively)

$$(2) \qquad a^i\, b^j\, x, \ \text{ for } i, j \geq 0,\ i + j \leq 2m - 1, \qquad \text{and} \qquad a^i\, b^j\, y, \ \text{ for } i, j \geq 0,\ i + j \leq m - 1.$$

(So we have $2m(2m+1)/2 = 2m^2 + m$ basis elements $a^i\, b^j\, x$, and $m(m+1)/2 = (m^2 + m)/2$ basis elements $a^i\, b^j\, y$, totaling $(5m^2 + 3m)/2$ elements.)

We now define four linear maps, a, b, c and d on M. Of these, a and b act in the obvious ways on the elements $a^i\,b^j\,x$ with $i+j<2m-1$ and on the elements $a^i\,b^j\,y$ with $i+j<m-1$, namely, by increasing the formal exponent of a, respectively, b, by 1; while they annihilate the elements for which these formal exponents have their maximum allowed total value, $i+j=2m-1$, respectively $i+j=m-1$. On the other hand, we let c annihilate x, but take y to $a^m\,x$, and hence (as it must if it is to commute with a and b) take $a^i\,b^j\,y$ to $a^{m+i}\,b^j\,x$, and we similarly let d annihilate x, but take y to $b^m\,x$, and hence $a^i\,b^j\,y$ to $a^i\,b^{m+j}\,x$.

It is easy to verify that these four linear maps commute, and are nilpotent. I claim that the unital algebra that they generate has for a basis the elements

$$(3)\qquad a^i\,b^j \text{ for } i,j\geq 0,\ i+j\leq 2m-1, \qquad\text{and}\qquad a^i\,b^j\,c \text{ and } a^i\,b^j\,d \text{ for } i,j\geq 0,\ i+j\leq m-1$$

(compare with (2)). Indeed, it is immediate that every monomial in a, b, c and d other than those listed in (3) is zero. Now suppose some k-linear combination of the monomials (3) were zero. By applying that linear combination to $x\in M$, we see that the coefficients in k of the monomials $a^i\,b^j$ (with no factor c or d) are all zero. Applying the same element to y, and noting that the sets of basis elements $a^{m+i}\,b^j\,x$ $(i+j\leq m-1)$ and $a^i\,b^{m+j}\,x$ $(i+j\leq m-1)$ are disjoint, we conclude that the coefficients of the monomials $a^i\,b^j\,c$ and $a^i\,b^j\,d$ are also zero.

Counting the elements (3), we see that the dimension of our algebra is $(2m^2+m)+(m^2+m)/2+(m^2+m)/2=3m^2+2m$, and this exceeds that of M by $(m^2+m)/2$, which is unbounded as m grows. The limit as $m\to\infty$ of the ratio of the dimensions of A and M is $\lim\,(3m^2+2m)/((5m^2+3m)/2)=6/5$.

Can we get a similar example with limiting ratio $5/4$, the ratio occurring for our 4-dimensional M? In fact we can. The description is formally like that of the above example, but with the basis (2) replaced by the slightly more complicated basis,

$$(4)\qquad a^i\,b^j\,x,\ \text{ where } 0\leq i,j<2m \text{ and } \min(i,j)<m, \qquad\text{and}\qquad a^i\,b^j\,y,\ \text{ where } 0\leq i,j<m.$$

We again define a and b to act by adding 1 to the relevant exponent symbol when this leads to another element of the above basis, while taking basis elements to zero when it does not; and we again let c and d annihilate x, but carry y to $a^m\,x$, respectively, $b^m\,x$, and act on other basis elements as they must for our operators to commute. We find that $\dim_k M=4m^2$ while $\dim_k A=5m^2$, so indeed, $\dim_k A/\dim_k M=5/4$.

Incidentally, the $m=1$ case of both the construction using (2) and the one using (4) can be seen to be isomorphic to the 5-dimensional algebra of 4×4 matrices with which we began this section.

4. The recalcitrant 3-generator question

It is not known whether every 3-generator commutative k-algebra A of endomorphisms of a finite-dimensional k-vector-space V satisfies $\dim_k A\leq\dim_k V$. In leading up to the discussion of that question, let me start with some observations applicable to any commutative algebra A of endomorphisms of a finite-dimensional vector space V. We will again write M for V regarded as an A-module. Note that since A is an algebra of endomorphisms of M, it acts faithfully on M.

Since A is a finite-dimensional commutative k-algebra, it is a direct product of local algebras, and the idempotents arising from the decomposition of $1\in A$ yield a corresponding decomposition of M as a direct product of modules over one or another of these factors. The question of whether, under given conditions, $\dim_k A$ can exceed $\dim_k M$ typically reduces to the corresponding question for those local algebras; so we shall assume A local in what follows. Moreover, since passing to the algebraic closure of k does not affect the properties we are interested in, we can assume that k is algebraically closed, hence that the residue field of the local algebra A is k itself. (Having reduced our considerations to this case, we can now drop the hypothesis that k be algebraically closed, keeping only this condition on the residue field.) Thus, the k-algebra structure of A is determined by that of its maximal ideal, which, by finite-dimensionality, is nilpotent. We shall denote this ideal $\mathbf{m}$.

Next note that if M is cyclic as an A-module, then it will be isomorphic as an A-module to A itself. (This is a consequence of commutativity. If $M=A\,x$, and some nonzero element of A annihilated x, then by commutativity, it would annihilate all of M, contradicting the assumption that A acts faithfully.)

Note also that the dual space $M^*=\mathrm{Hom}_k(M,\,k)$ acquires a natural structure of A-module (since the vector-space endomorphisms of M given by the elements of A induce endomorphisms of M^*), of the same

k-dimension as M. Using this duality, it follows from the preceding observations that $\dim_k A = \dim_k M$ will also hold if M^* is cyclic. The latter condition is equivalent to saying that the socle of M (the annihilator of $\mathbf{m}$ in M) is 1-dimensional; in this situation one calls M "cocyclic". (Dually, the condition that M be cyclic is equivalent to saying that $\mathbf{m}M$ has codimension 1 in M.)

Thus, if $\dim_k A$ and $\dim_k M$ are to be distinct, M can be neither cyclic nor cocyclic. For brevity of exposition, let us focus on the consequences of M being noncyclic.

The most simpleminded way to get a noncyclic module M is to take a direct sum of two nonzero cyclic modules. (In doing so, we keep the assumption that A acts faithfully on M, though it will not in general act faithfully on these summands.) In this situation, I claim that A will have dimension *strictly smaller* than that of M. Indeed, writing the direct summands as $M_1 \cong A/I_1$ and $M_2 \cong A/I_2$ for proper ideals $I_1,\ I_2 \subseteq A$, we see that the k-dimension of M is the sum of the codimensions of I_1 and I_2 in A, while the dimension of A is the codimension of the zero ideal, which in this case is $I_1 \cap I_2$. In view of the exact sequence $0 \to A/(I_1 \cap I_2) \to A/I_1 \oplus A/I_2 \to A/(I_1 + I_2) \to 0$, the dimension of the direct sum M must exceed that of the algebra A by $\dim_k A/(I_1 + I_2)$, which is positive because I_1 and I_2 lie in the common ideal $\mathbf{m}$.

Hence, if we want a module M such that

$$\dim_k A \ > \ \dim_k M, \tag{5}$$

with M generated by two elements x_1 and x_2, so that it is a homomorphic image of $A\,x_1 \oplus A\,x_2$, then the construction of this homomorphic image must involve additional relations, i.e., the identification of a nonzero submodule of $A\,x_1$ with an isomorphic submodule of $A\,x_2$.

It turns out that a single relation, i.e., the identification of a *cyclic* submodule of $A\,x_1$ with an isomorphic cyclic submodule of $A\,x_2$, is still not enough to get (5). For let the common isomorphism class of the cyclic submodules that we identify be that of A/I_3. Then I_1 and I_2 are both contained in I_3, hence so is $I_1 + I_2$. Now we have seen that the amount by which $\dim_k (A\,x_1 \oplus A\,x_2)$ exceeds $\dim_k A$ is the codimension of $I_1 + I_2$ in A; so setting to zero a submodule isomorphic to A/I_3, which has dimension at most that codimension, can at best give us equality.

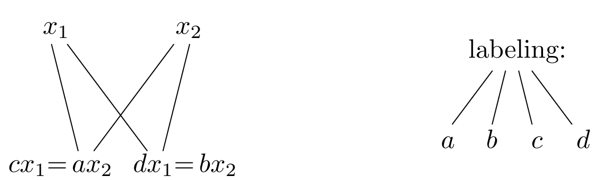
Thus, we need to divide out by at least a 2-generator submodule to get (5). And indeed, the families of 4-generator examples we obtained in the latter half of the preceding section can be thought of as constructed by imposing two relations, $cy = a^m x$ and $dy = b^m x$, on a direct sum $A\,x \oplus A\,y \cong A/I_1 \oplus A/I_2$.

In the remainder of this section, I will display a few examples diagrammatically. In these examples, M will have a k-basis B such that each of our given generators of A carries each element of B either to another element of B or to 0. The actions of the various generators of A on basis elements will be shown as downward line segments of different slopes, with the matching of generator and slope shown to the right of the diagram (under the word "labeling"). Where no line segment of a given slope descends from a given element, this means that the corresponding generator of A annihilates that basis element. For instance, the 4-dimensional matrix ring example with which we began the preceding section may be diagrammed

(6)

The fact that it can be obtained from a direct sum of two cyclic modules $A\,x_1,\ A\,x_2$ by two identifications is made clear in the representation below, where the labels on the lower vertices show the relations imposed.

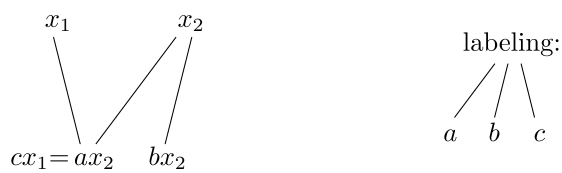
(7)



(In (7), I have changed the edge-labeling from (6), since that labeling, based on the use of upper triangular matrices in the preceding section, would have led to lower-indexed basis elements on the bottom and higher-indexed basis elements on the top, and hence, when the former were replaced by expressions in the latter, a notation where everything was expressed in terms of elements apparently arbitrarily called x_3 and x_4.)

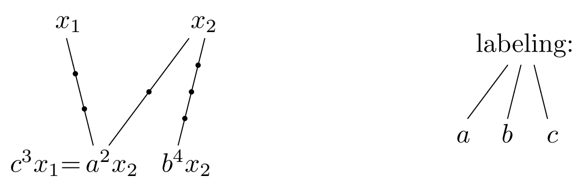
If we drop one of the generators of the above A, getting a 3-generator algebra, then the same vector space becomes, as shown in the next diagram, a module whose submodules $A\,x_1$ and $A\,x_2$ are connected by only one relation, and which has $\dim_k A = \dim_k M$ (both dimensions being 4), which, we have seen, is the best one can hope for from such a "one-relation" module.

(8)



We remark that we still get equality of dimensions if, in the above example, we replace one or more of the 1-step paths by multistep paths repeating the algebra generator in question; e.g.,

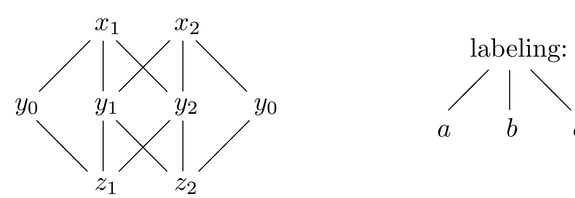
(9)



which has $\dim_k A = \dim_k M = 10$.

I have attempted to find examples of 3-generator algebras A such that $\dim_k A > \dim_k M$, by connecting two or more cyclic modules with the help of two or more relations, but without success; the best my fiddling with such examples has achieved is to get more examples of equality; for instance,

(10)



Note the repetition of y_0 at the right end of the middle row; thus, y_0 is both $a\,x_1$ and $c\,x_2$. We find that the distinct nonzero monomials in A are

$$1;\ a,\ b,\ c;\ a^2 = bc,\ b^2 = ac,\ c^2 = ab, \tag{11}$$

which are linearly independent, so that A, like M, is 7-dimensional. (One must also verify commutativity of A. This is fairly easy; there are three relations to be checked, $ab = ba$, $ac = ca$ and $bc = cb$, and these need only be checked on x_1 and x_2, since on all other basis elements, both sides of each equation clearly give 0.)

One can modify this example by making the rightmost basis-element in the middle row be, not a repetition of $y_0 = ax_1$, but $a^i y_0 = a^{i+1}x_1$, for any $i > 0$. This adds exactly i basis elements to M, namely $a\,y_0, \dots, a^i y_0$, and likewise adds i monomials to the basis of A, namely $a^3, \dots, a^{i+2}$ (with a^{i+2} rather than a^2 now coinciding with bc). In the new A, the relation $c^2 = ab$ no longer holds; rather, $c^2 = 0$, but ab remains nonzero.

We see that this modified example still satisfies $\dim_k A = \dim_k M$. In fact, Kevin O'Meara has pointed out to me that by [14, Theorem 5.5.8], if a k-algebra A of endomorphisms of a k-vector-space M generated by three commuting elements a, b, c is to satisfy $\dim_k A > \dim_k M$, then M must require at least 4 generators as a module over the subring $k[a]$. (The wording of that theorem is that $\dim_k A \leq \dim_k M$ holds for "3-regular" matrix algebras, i.e., those where M can be so generated by 3 elements.) The example of (10), and the variant just noted, are generated over $k[a]$ by $\{x_1, x_2, y_2\}$ confirming that we would need something more complicated to get a counterexample. O'Meara suspects that the analog of the theorem just quoted also holds for 4-regular algebras, but might fail in the 5-regular case. (Cf. [14, p.226, footnote 12].) Incidentally, (9) is an example of a module that is not 3-generated over any of $k[a]$, $k[b]$, $k[c]$, but which still satisfies $\dim_k A = \dim_k M$.

We remark, for the benefit of the reader who wants to explore examples using diagrams like the above, that a consequence of our requirement of commutativity is that wherever the diagram shows distinct generators of A coming "into" and "out of" a vertex, e.g., , these must be part of a parallelogram . So, for instance, if we tried to improve on (10) by deleting the line segment from y_2 to z_2, so that b^2, though still nonzero because of its action on x_1, ceased to equal ac, thus increasing $\dim_k A$ by 1, the resulting algebra would not be commutative, because the configuration consisting of x_1, y_1 and z_2 and their connecting line segments would not be part of a parallelogram. (On the other hand, instances of or of do not need to belong to parallelograms, as illustrated by x_1, x_2, y_2 in (10).)

Our diagrammatic notation also allows us to illustrate the fact mentioned in §2, that a faithful module over a finite-length homomorphic image A of $k[s,t]$ need not contain an isomorphic copy of A as a $k[s,t]$-module, though it will as a $k[s]$-module. Let $A = k[s,t]/(s,t)^2$, which has diagram . Let M be the dual module $\mathrm{Hom}_k(A,k)$. This is faithful, but has diagram , which does not contain a copy of the diagram of A. However, the diagrams for A and M as $k[s]$-modules are and , which are isomorphic.

5. Some questions, and steps toward their answer

Theorem 3 has the unsatisfying feature that our R has absorbed one of the indeterminates of the original polynomial algebra $k[s,\, t]$, but not the other. We may ask, without referring to indeterminates,

Question 4. *For which commutative rings S does the statement,*

(12) *For every S-module M of finite length, if we let $A = S/\mathrm{Ann}_S\, M$, then* $\mathrm{length}_S(A) \leq \mathrm{length}_S(M)$,

hold?

Theorem 3 says roughly that the class of such rings includes the rings $R[t]$ where R is a principal ideal domain. A plausible generalization would be the statement that it contains all rings S such that every maximal ideal $\mathbf{m}$ of S satisfies $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$. In fact, if Gerstenhaber's result should turn out to go over to 3-generator algebras of commuting matrices, then we can hope that (12) will hold for all S whose maximal ideals satisfy $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 3$.

(There is a slight difficulty with regarding (12) as a generalization of the property of Theorem 3. When $S = R[t]$, Theorem 3 concerns S-modules of finite length over R, while (12) concerns S-modules of finite length over S, and these are not always the same. For instance, if R is a discrete valuation ring with maximal ideal (p), then the $R[t]$-module $R[t]/(pt-1)$ has length 1 as an $R[t]$-module, since the ring $R[t]/(pt-1)$ is a field, but has infinite length as an R-module. Since "length over R" has no meaning for a module over a ring S that is not assumed to be built from a subring R, we shall take condition (12) as our focus from here on.)

Note that a commutative ring S satisfies (12) if and only if all of its finite-length homomorphic images A do; equivalently, if and only if all those images A have the stated property for *faithful* A-modules M. Now for a ring, being of finite length is equivalent to being Artinian, and every commutative Artinian ring is a finite direct product of local rings. This leads to the modified question,

Question 5. *For which commutative Artinian local rings A does the statement*

(13) *Every faithful A-module M satisfies* $\mathrm{length}_A(M) \geq \mathrm{length}_A(A)$

hold?

We can get further mileage on these questions by combining Theorem 3 with some theorems of I. S. Cohen [4]. (Note to the reader of that paper: a "local ring" there means what is now called a Noetherian local ring. Since the local rings we apply Cohen's results to will be Artinian, this will be no problem to us. Incidentally, Cohen defines a *generalized* local ring to mean what we would call a (not necessarily Noetherian) local ring whose maximal ideal $\mathbf{m}$ is finitely generated and satisfies $\bigcap \mathbf{m}^n = \{0\}$, and he comments that he does not know whether every such ring is "local", i.e., is also Noetherian. This has been answered in the negative [9].)

Recall that a local ring A with maximal ideal $\mathbf{m}$ is said to be *equicharacteristic* if the characteristics of A and $A/\mathbf{m}$ are the same. This is equivalent to saying that A contains a field. (The implication from "contains a field" to "equicharacteristic" is clear. Conversely, note that since $A/\mathbf{m}$ is a field, its characteristic is 0 or a prime number p. In the former case, every member of $\mathbb{Z}-\{0\}$ is invertible in $A/\mathbf{m}$, and hence in A, so A contains the field $\mathbb{Q}$; while in the latter, if A is equicharacteristic, then, like $A/\mathbf{m}$, it has characteristic p, and so contains the field $\mathbb{Z}/(p)$.)

Cohen shows in [4, Theorem 9, p. 72] that a complete Noetherian local ring which is equicharacteristic is a homomorphic image of the ring of formal power series in $\mathrm{length}(\mathbf{m}/\mathbf{m}^2)$ indeterminates over a field, where $\mathbf{m}$ is the maximal ideal of the ring. Using this, we can get

Proposition 6. *Suppose A is a commutative local Artinian ring with maximal ideal $\mathbf{m}$, and that $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$. Then if A is equicharacteristic, it satisfies* (13).

Hence, if S is a commutative ring such that every maximal ideal $\mathbf{m} \subseteq S$ satisfies $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$, and S contains a field, then S satisfies (12).

Proof. We shall prove the first assertion. Clearly, the second will then follow by applying the first to local factor-rings of S.

Since the local ring A is Artinian, it is complete, so by the result of Cohen's cited, it is a homomorphic image of a formal power series ring in ≤ 2 indeterminates over a field. But a finite-length homomorphic image of a formal power series ring is an image of the corresponding polynomial ring. Hence by the result of Gerstenhaber with which we started, A satisfies (13). $\square$

Cohen's result for mixed characteristic is [4, Theorem 12, p. 84]. The case we shall use, that of the last sentence of that theorem, says that if A is a complete Noetherian local ring whose residue field $A/\mathbf{m}$ has characteristic p (i.e., such that $p \in \mathbf{m}$), but such that $p \notin \mathbf{m}^2$, then A can be written as a homomorphic image of a formal power series ring in $\mathrm{length}(\mathbf{m}/\mathbf{m}^2)-1$ indeterminates over a complete discrete valuation ring V in which p has valuation 1. (Intuitively, p takes the place of one of the indeterminates in the result for the equicharacteristic case.) This gives us

Proposition 7. *Again let A be a commutative local Artinian ring with maximal ideal $\mathbf{m}$, such that $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$. If $p \in \mathbf{m}$ but $p \notin \mathbf{m}^2$, then A satisfies* (13).

Hence, if S is a commutative ring such that every maximal ideal $\mathbf{m} \subseteq S$ satisfies $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$, and such that no prime $p \in \mathbb{Z}$ belongs to the square of any maximal ideal of S, then S satisfies (12).

Proof. We will prove the first assertion. The second will then follow by applying that assertion to local factor rings whose residue fields have prime characteristic, while applying the first assertion of Proposition 6 to local factor rings whose residue fields have characteristic zero.

In the situation of the first assertion, the result of Cohen cited, again combined with the observation that a finite-length homomorphic image of a formal power series ring is a homomorphic image of the corresponding polynomial ring, tells us that A is a homomorphic image of a polynomial ring in at most one indeterminate over a discrete valuation ring V. Hence by Theorem 3 above, A satisfies (13). $\square$

If, in the mixed-characteristic case, we instead have $p \in \mathbf{m}^2$, Cohen's result only tells us that A is a homomorphic image of a formal power series ring in $\mathrm{length}(\mathbf{m}/\mathbf{m}^2)$ (rather than $\mathrm{length}(\mathbf{m}/\mathbf{m}^2)-1$) indeterminates over a complete discrete valuation ring V; so in our case, A is a homomorphic image of $V[[s,t]]$. In general, this is not enough to give us the conclusion we want, but there are cases where it is. Let d be the integer such that $p \in \mathbf{m}^d - \mathbf{m}^{d+1}$, and suppose that

(14) p has a d-th root q in A.

Then this d-th root q will lie in $\mathbf{m}-\mathbf{m}^2$, so via a change of variables, the indeterminate s in $V[[s,t]]$ may be taken to be an element that maps to $q \in A$. Thus, A is a homomorphic image of $V[[s,t]]/(s^d-p)$; so using, as before, the fact that A has finite length, we see that A is in fact a homomorphic image of $V[s,t]/(s^d-p)$. But $V[s]/(s^d-p)$ is a discrete valuation ring $V' \supseteq V$, so A is a homomorphic image of $V'[t]$, and we can again conclude from Theorem 3 that it satisfies (13).

Can we generalize this further? It might seem harmless to weaken (14) to say that some *associate* of p in A has a d-th root $q \in A$. But then the problem arises of where the unit of A that carries p to q^d lies.

If it does not belong to the image of V, we can't use it in constructing our extension V'. We might hope to incorporate the condition that that unit lie in the image of V into a generalization of condition (14); but a version of Proposition 7 based on such a condition would be awkward to formulate, since the V given by Cohen's result is not part of the hypothesis of Proposition 7. One assumption that *will* clearly guarantee that we can argue as suggested is that the unit in question lie in the image of $\mathbb{Z}$ in A. I will not try here to find the "best" result of this sort.

If we don't assume any condition like (14), there are examples where A indeed cannot be generated by one element over a homomorphic image of a discrete valuation ring. For instance, let p be any prime, and within $\mathbb{Z}[p^{1/5}]$, let us take the subring $\mathbb{Z}[p^{2/5}, p^{3/5}]$ and divide out by the ideal (p^2), writing

$$A \;=\; \mathbb{Z}[p^{2/5}, p^{3/5}]/(p^2). \tag{15}$$

We see that A is local and Artinian, with maximal ideal $\mathbf{m}$ generated by $\{p^{2/5}, p^{3/5}, p\}$; and since the last of these elements is the product of the first two, $\mathbf{m}$ is in fact 2-generated, and $\mathbf{m}/\mathbf{m}^2$ can be seen to have length 2. But I claim that A is not 1-generated over a homomorphic image B of a valuation ring V. Roughly speaking, if it were, then that subring $B \subseteq A$ would either have to have the property that all its elements are associates of powers of $p^{2/5}$, or that they are associates of powers of $p^{3/5}$; but $p \in B$ cannot be either.

Nevertheless, I would be surprised if the ring (15) did not satisfy (13). Any way I can think of to construct a candidate counterexample could be duplicated over $k[s^2, s^3]/(s^{10})$ for k a field, though we know that no counterexample exists in that case by Gerstenhaber's original result.

We can in fact show that for *all but finitely many* primes p, the ring (15) does satisfy (13). For suppose we had counterexamples for an infinite set P of primes. Let us write A_p $(p \in P)$ for the corresponding rings (15), and choose for each $p \in P$ an A_p-module M_p witnessing the failure of (13). Now let A be an ultraproduct of the A_p with respect to a nonprincipal ultrafilter on P, and M the corresponding ultraproduct of the M_p, an A-module. From the fact that the A_p all have the same length (namely 10), one can verify that A will also have that length, hence be Artinian, and from the fact that $\mathrm{length}\,\mathbf{m}/\mathbf{m}^2 = 2$ for all A_p, one finds that the same is true for A. Moreover, the characteristic of $A/\mathbf{m}$ will be 0, because every prime integer is invertible in all but one of the $A_p/\mathbf{m}_p$; hence A is necessarily equicharacteristic. The ultraproduct M will be a faithful A-module, and since by assumption all the M_p have lengths less than the common length of the A_p, the module M will also have length less than that common value. Hence M witnesses the failure of (13) for A, contradicting Proposition 6; so there cannot be such an infinite set P of primes.

We see that the above method of reasoning in fact gives

Proposition 8. *For every positive integer n, there are at most finitely many primes p for which there exist commutative local Artinian rings A of length n and characteristic a power of p which satisfy* $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 2$, *but fail to satisfy* (13). ☐

Above, we have, for brevity, been focusing on the more challenging aspects of our problem. One can also formally extend our results in more trivial ways. For instance, using the case of Cohen's [4, Theorem 12, p. 84] that does not make the assumption $p \notin \mathbf{m}^2$ (quoted following Proposition 7 above), we see that any A having $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 1$ satisfies (13), with no need for a condition on the behavior of integer primes p. Also, one can easily extend the final assertion of Proposition 6 to a commutative ring S which, rather than containing a field, contains a direct product of fields, or more generally, a von Neumann regular subring. Still more generally, using the first statements of both those propositions, we can extend the second statements thereof to rings S such that for every maximal ideal $\mathbf{m} \subseteq S$ and prime $p \in \mathbb{Z}$, either $p \notin \mathbf{m}^2$ or $p \in \bigcap_n \mathbf{m}^n$.

Let us now turn to rings S and A for which we can show that (12) or (13) does *not* hold. The first assertion of the next result generalizes our observations on the algebra described in (2) and (3). (This will be clearer from the proof than from the statement.)

Proposition 9. *Suppose A is a commutative local Artinian ring, with maximal ideal* $\mathbf{m}$. *If A has ideals I_1 and I_2 with zero intersection, such that A/I_1 and A/I_2 have isomorphic submodules $J_1/I_1 \cong J_2/I_2$ satisfying*

$$\mathrm{length}(A/I_1) \,+\, \mathrm{length}(A/I_2) \,-\, \mathrm{length}(J_1/I_1) \,<\, \mathrm{length}(A), \tag{16}$$

(*equivalently,* $\mathrm{length}(A) < \mathrm{length}(J_1) + \mathrm{length}(I_2)$), *then* A *does not satisfy* (13).

In particular, this is the case if A *is any commutative local Artinian ring satisfying* $\mathbf{m}^2 = \{0\}$ *and* $\mathrm{length}(\mathbf{m}) \geq 4$.

Hence, no commutative ring S *having a maximal ideal* $\mathbf{m}$ *with* $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \geq 4$ *satisfies* (12). (*In this last statement, we do not require* $\mathrm{length}(\mathbf{m}/\mathbf{m}^2)$ *to be finite.*)

Proof. In the situation of the first paragraph, let M be the module obtained from $A/I_1 \oplus A/I_2$ by identifying the isomorphic submodules $J_1/I_1 \subseteq A/I_1$ and $J_2/I_2 \subseteq A/I_2$. Each of A/I_1 and A/I_2 still embeds in M, so since the annihilators I_1 and I_2 of these modules have zero intersection, M is a faithful A-module. Now $\mathrm{length}(M)$ is given by the left-hand side of (16), hence that inequality shows the failure of (13). The parenthetical statement of equivalence on the line after (16) is seen by expanding the expressions of the form "$\mathrm{length}(P/Q)$" in (16) as $\mathrm{length}(P) - \mathrm{length}(Q)$, and simplifying.

(In the example described in (2) and (3), we can take $I_1 = \mathrm{Ann}_A(x) = (c, d)$, $I_2 = \mathrm{Ann}_A(y) = (a^m, a^{m-1}b, \ldots, b^m)$, $J_1 = \{f \in A \mid fx \in Acy + Ady\} = I_1 + (a^m, b^m)$, and $J_2 = \{f \in A \mid fy \in Acy + Ady\} = I_2 + (c, d)$.)

To get the assertion of the second paragraph, let $\mathrm{length}(\mathbf{m}) = d$, so that $\mathbf{m}$ can be regarded as a d-dimensional vector space over $A/\mathbf{m}$. Let I_1 and I_2 be any subspaces of $\mathbf{m}$ of equal dimension $e \geq 2$, and having zero intersection (these exist because $d \geq 4$), and let $J_1 = J_2 = \mathbf{m}$. By comparison of dimensions, $J_1/I_1 \cong J_2/I_2$ as $A/\mathbf{m}$-modules, and hence as A-modules. Now $\mathrm{length}(J_1) + \mathrm{length}(I_2) = d + e > d + 1 = \mathrm{length}(A)$, giving the inequality noted parenthetically as equivalent to (16).

For S and $\mathbf{m}$ as in the final statement, let A_0 be the local ring $S/\mathbf{m}^2$, with square-zero maximal ideal $\mathbf{m}_0 = \mathbf{m}/\mathbf{m}^2$. Since A_0 need not have finite length, let us divide out by an $A_0/\mathbf{m}_0$-subspace of $\mathbf{m}_0$ whose codimension is finite but ≥ 4. The result is a homomorphic image A of S which has finite length and, by the second assertion of the lemma, fails to satisfy (13). Hence S fails to satisfy (12). □

If it should turn out that (13) holds for every A with $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 3$, we would have a complete answer to Question 4; for comparing that fact with Proposition 9, we could conclude that the rings S satisfying (12) are precisely those for which all maximal ideals $\mathbf{m}$ satisfy $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \leq 3$.

From the proof of Proposition 9, we can see that the existence of ideals satisfying (16) is necessary and sufficient for the existence of a 2-*generator* A-module M witnessing the failure of (13). For higher numbers of generators, it seems hard to formulate similar necessary and sufficient conditions; though one can give *sufficient* conditions, corresponding to necessary and sufficient conditions for the existence of such modules with particular sorts of structures (e.g., sums of three cyclic submodules, each pair of which is glued together along a pair of isomorphic submodules), and these might be useful in looking for examples.

We have seen that the algebras described in §3 are cases of Proposition 9. For a further example, suppose we adjoin to $\mathbb{Z}$ the 7-th root of a prime p, and then pass to the subring

$$S \;=\; \mathbb{Z}\,[p^{4/7},\, p^{5/7},\, p^{6/7}]. \tag{17}$$

This has a maximal ideal $\mathbf{m}$ generated by $\{p^{4/7}, p^{5/7}, p^{6/7}, p\}$, and these generators are linearly independent modulo $\mathbf{m}^2$, so by Proposition 9, S does not satisfy (12).

The next result, and the discussion that follows it, will give us further classes of rings A that *do* satisfy (13). However, these classes are not closed under homomorphic images, and can fail to satisfy (12). Thus, though the results will add to what we know regarding Question 5, they say little about Question 4, which inspired that question.

We recall that a commutative local Artinian ring A is said to be *Frobenius* if it is cocyclic as an A-module, i.e., if its socle has length 1. (For an Artinian but not-necessarily-commutative, not-necessarily-local ring, the Frobenius condition is the statement that the socle is isomorphic as right and as left module to $A/J(A)$ [13, Theorem 16.14(4)].)

Lemma 10. *Every Frobenius commutative local Artinian ring* A *satisfies* (13).

Proof. If M is a faithful A-module, then M has an element x not annihilated by $\mathrm{socle}(A)$. Since $\mathrm{socle}(A)$ is simple, the annihilator of x has trivial intersection with that socle, hence is zero. So $A\,x$ is a faithful cyclic A-module, hence has length equal to the length of A, so $\mathrm{length}\,M \geq \mathrm{length}\,A$. □

For an example of a ring as in the above lemma which has $\mathrm{length}(\mathbf{m}/\mathbf{m}^2) \geq 4$, and therefore, though we have just seen that it satisfies (13), will not satisfy (12), let k be a field, take any $n_1, \ldots, n_4 > 0$, and let

$A = k[t_1, t_2, t_3, t_4]/(t_1^{n_1+1}, t_2^{n_2+1}, t_3^{n_3+1}, t_4^{n_4+1})$. Then the socle of A is the 1-dimensional space spanned by the element $t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4}$, so A is Frobenius, but $\mathbf{m}/\mathbf{m}^2$ is 4-dimensional, with basis t_1, t_2, t_3, t_4.

Let us also note, in contrast with the above lemma, that a *large* socle does not *prevent* a ring from satisfying (13). For instance, for k a field and n any positive integer, the algebra $A = k[s, t]/(s^n, s^{n-1}t, \ldots, s\,t^{n-1}, t^n)$ has socle of length n, with basis $\{s^{n-1}, s^{n-2}\,t, \ldots, t^{n-1}\}$, but by Proposition 6, A satisfies (13).

In an earlier version of this note, after proving Lemma 10, I asked whether the condition that socle(A) have length ≤ 2, or perhaps even ≤ 3, might imply (13). In response, Luchezar Avramov pointed me to a result of T. Gulliksen [6, Theorem 1] which indeed proves this for length(socle$(A)) \leq 3$. Thinking about Gulliksen's method eventually resulted in [3], where a lemma used by Gulliksen [6] is strengthened in several ways. In particular, one gets the following fact, which may be useful in the context of the present note.

(18) (Case of [3, Theorem 3]) If A is a commutative Artinian ring with maximal ideal $\mathbf{m}$, and M a faithful A-module of minimal length, then $\mathrm{length}(M/\mathbf{m}\,M) + \mathrm{length}(\mathrm{socle}(M)) \leq \mathrm{length}(\mathrm{socle}(A)) + 1$.

6. Other sorts of questions

Though the focus of this note has been the condition $\mathrm{length}\,A \leq \mathrm{length}\,M$, one can ask, more generally, how big $\mathrm{length}\,A/\,\mathrm{length}\,M$ can become in cases where it exceeds 1. To maximize the hope of positive results, I will pose the question here for algebras of endomorphisms of vector spaces.

Question 11. *For each positive integer d, let r_d be the supremum of the ratio $\dim_k A/\dim_k V$, over all commutative* d-generator *algebras A of endomorphisms of nonzero finite-dimensional vector-spaces V over arbitrary fields k.*

(*Thus, the r_d form a nondecreasing sequence, whose terms are real numbers or $+\infty$. We know that $r_1 = r_2 = 1$, and the examples of* §4 *and* §3 *respectively suggest that r_3 may be 1, and r_4 may be 5/4.*)

Determine as much as possible about this sequence. In particular, are all its terms finite?

I don't even see a proof that r_3 finite! (If we did not restrict ourselves to *commuting* endomorphisms, these suprema would be infinite for all $d \geq 2$, since the full $n \times n$ matrix algebra can be generated by two matrices, and the ratio of its dimension to that of the space on which it acts, $n^2/n = n$, is unbounded.)

Something we *can* say is that as a function of d, the r_d increase without bound:

Lemma 12. *If d, e_0, e_1 are positive integers such that $d \geq e_0\,e_1$, then $r_d \geq (e_0\,e_1 + 1)/(e_0 + e_1)$.*

Hence, taking $e_0 = 2m - 1$, $e_1 = 2m + 1$, we see that $r_{4m^2-1} \geq m$.

Proof. Let V be the direct sum of an e_0-dimensional vector space V_0 and an e_1-dimensional space V_1, and A be the algebra of endomorphisms of V spanned by the identity, and all endomorphisms that carry V_0 into V_1 and annihilate V_1. Since any two endomorphisms of the latter sort have product 0, A is commutative. It is generated as an algebra by any basis of the $e_0 e_1$-dimensional space of such endomorphisms, hence *a fortiori* it can be generated by $d \geq e_0\,e_1$ elements. Since A has dimension $e_0\,e_1 + 1$ and V has dimension $e_0 + e_1$, we get $r_d \geq (e_0\,e_1 + 1)/(e_0 + e_1)$, as claimed.

The final sentence clearly follows. □

In the spirit of [2, §3], we might expect that the inequalities we have obtained for algebras of endomorphisms of vector spaces would entail analogous inequalities for monoids of endomaps of sets, with cardinalities replacing dimensions. But this is not the case. For instance, a 1-generator group of permutations of an n-element set can have order much larger than n, if the generating permutation has many cycles of relatively prime lengths. (The reason why results on algebras don't imply the corresponding results for monoids is that the matrices corresponding to a family of distinct endomaps of a finite set need not be linearly independent.)

I will end by repeating, in slightly generalized form, a question I asked in [2], which resembles the subject considered *here* (and differs from the subject considered *there*) in that it asks whether the size of a certain family of actions is bounded by the size of the object it acts on; but which is otherwise only loosely related to the topic of either paper.

Question 13 (after [2, Question 23])**.** *Let R be a commutative algebra over a commutative ring k, let V be a k-submodule of R, and let n be a positive integer such that the k-submodule $V^n \subseteq R$ of all sums of n-fold products of elements of V has finite length as a k-module. Then must*

(19) $$\mathrm{length}_k\,(V/\mathrm{Ann}_V(V^n)) \;\leq\; \mathrm{length}_k(V^n),$$

where $\mathrm{Ann}_V(V^n)$ *denotes* $\{x \in V \mid x\,V^n = \{0\}\} \subseteq V$ *?*

7. Acknowledgements

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